SCET at Next-to-Leading Power

(Matthias Neubert, 28 February 2026)

Outline:

- Motivation
- SCET Lagrangian at NLP
- $h \to \gamma\gamma$ as a Study Case
- Systematic Removal of Endpoint Divergences
- General Lessons

Suggested literature:

SCET review: T. Becher, A. Broggio, A. Ferroglia: 1410.1892

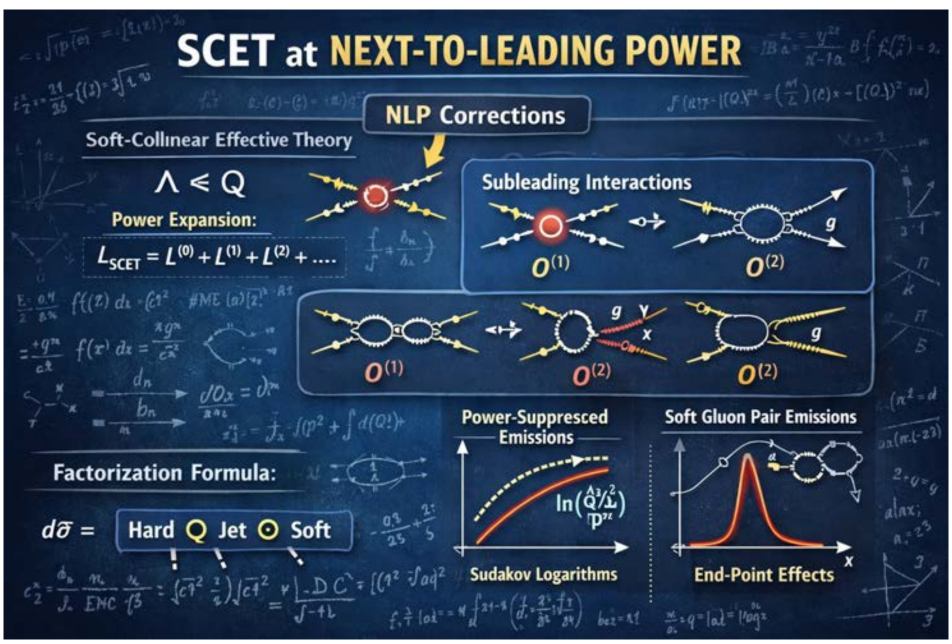
Papers on which the lectures are based:

M. Beneke, A.P. Chapovsky, M. Diehl, T. Feldmann: hep-ph/0206152

D. Pirjol, I.W. Stewart: hep-ph/0211251

→ Z.L. Liu, M. Neubert: 1912.08818

Z.L. Liu, B. Mecaj, M. Neubert, X. Wang: 2009.06779



(ChatGPT)

I. Motivation

As a well-defined effective field theory (EFT), SCET should provide a systematic scale separation at all orders in perturbation theory and all orders in the expansion parameter $\lambda \ll 1$.

While for most other EFTs the extension to next-to-leading power (NLP) is "straightforward but tedious" (e.g. χPT, HQET, SMEFT), for SCET major complications appear in this step, whose handling require new conceptual advances:

- endpoint-divergent convolution integrals (familiar from QCD factorization at NLP)
- generalized Sudakov RGEs

Technical breakthrough:

- refactorization-based subtraction (RBS) scheme

Controlling SCET beyond the leading power in λ is thus of conceptual importance!

But it is also of practical relevance:

1) The amplitude for some processes are intrinsically power suppressed, so that NLP SCET is required to describe them.

e.g.: $h \to \gamma\gamma$ via b-quark loop,
QED corrections to $B^- \to \mu^- \bar{\nu}_\mu$ (2601.14361)
($\hookrightarrow$ chiral suppression)

2) For very accurate calculations, including NLP corrections may be necessary to reach the accuracy goal.

e.g.: N-jettiness slicing as a tool to perform IR subtraction in higher-order ($N^{n\geq 2}LO$) calculations in collider physics

II. SCET Lagrangian at NLP

Recall that in the construction of the SCET-1 Lagrangian the purely collinear Lagrangian:

$$\mathcal{L}_c = \bar{\xi}_n \frac{\not{\bar{n}}}{2}\, i n\cdot D_c\, \xi_n - \bar{\xi}_n\, i\not{D}_c^{\perp} \frac{\not{\bar{n}}}{2} \frac{1}{i\bar{n}\cdot D_c + i\delta}\, i\not{D}_c^{\perp}\, \xi_n$$

$$+ \text{(pure glue terms)}$$

is an exact rewriting of the QCD Lagrangian and does not receive any power corrections. Power corrections arise when we couple the collinear fields to a soft sector. The leading such interaction is:

$$\mathcal{L}_{c+us} = \bar{\xi}_n(x) \frac{\not{\bar{n}}}{2} \Big(\overset{\lambda^2}{i n\cdot\partial} + \overset{\lambda^2}{g_s\, n\cdot A_c(x)} + \overset{\lambda^2}{g_s\, n\cdot A_{us}(x_-)} \Big)\, \xi_n$$

$$+ \text{(analogous pure glue terms)} + \dots$$

Importantly, this interaction can be removed by the field redefinition ("decoupling transformation"):

new fields

$$\xi_n(x) \longrightarrow S_n(x_-)\, \xi_n^{(0)}(x)$$

$$A_c^{\mu}(x) \longrightarrow S_n(x_-)\, A_c^{\mu(0)}(x)\, S_n^{\dagger}(x_-)$$

where:

$$\boxed{S_n(x) = \mathcal{P}\exp\left(ig_s\int_{-\infty}^{0} dt\; n\cdot A_{us}(x+nt)\right)} \quad \text{(unitary)}$$

↳ basis of most SCET factorization theorems

SCET power counting:

$$\xi_n \sim \lambda\,, \qquad A_c^\mu \sim (\overset{+}{\lambda^2}, \overset{-}{1}, \overset{\perp}{\lambda})$$

$$q_{us} \sim \lambda^{3/2}\,, \qquad A_{us}^\mu \sim (\lambda^2, \lambda^2, \lambda^2)$$

same as momenta

shows that all other soft fields (except $n\cdot A_{us}$) are power-suppressed w.r.t. the corresponding collinear fields.

↳ generates series of higher-order terms in λ

Also, in interactions of soft fields with collinear field, the soft fields need to be multipole expanded:

$$x^\mu = \underbrace{n\cdot x\,\frac{\bar{n}^\mu}{2}} + \underbrace{\bar{n}\cdot x\,\frac{n^\mu}{2}} + x_\perp^\mu$$

$$\equiv \underset{1}{x_+^\mu} + \underset{\lambda^{-2}}{x_-^\mu} + \underset{\lambda^{-1}}{x_\perp^\mu} \quad \leftarrow \text{in interactions with collinear fields}$$

(since $x\cdot p_c \sim 1$)

$$\Rightarrow \quad \phi_{us}(x) = \phi_{us}(x_-) + \overset{\lambda^{-1}}{x_\perp} \cdot \overset{\lambda^{2}}{\partial_\perp} \, \phi_{us}(x_-)$$
$$+ \left(\underset{1}{x_+} \cdot \underset{\lambda^2}{\partial_-} + \underset{\lambda^{-2}}{\frac{x_\perp^\mu x_\perp^\nu}{2}} \, \underset{\lambda^2}{\partial^\perp_\mu} \, \underset{\lambda^2}{\partial^\perp_\nu} \right) \phi_{us}(x_-) + \dots$$

↳ generates series of higher-order terms in λ

Finally, in processes with heavy quarks, the use of HQET is another source of higher-order terms in λ.

The expansion of the SCET-1 Lagrangian to higher-order in λ in the coordinate formulation was performed in hep-ph/0206152 by Beneke, Chapovsky, Diehl and Feldmann, see eqs. (52)-(54): (see also: Pirjol, Stewart: hep-ph/0211251)

$$\mathcal{L} = \mathcal{L}_\xi^{(0)} + \mathcal{L}_\xi^{(1)} + \mathcal{L}_\xi^{(2)} + \mathcal{L}_{\xi q}^{(1)} + \mathcal{L}_{\xi q}^{(2)} + \mathcal{L}_{\rm us}$$

where:

$$\mathcal{L}_\xi^{(0)} = \bar\xi \left(i n_- D + i \not{D}_{\perp c} \frac{1}{i n_+ D_c} i \not{D}_{\perp c} \right) \frac{\not{n}_+}{2} \xi$$

$$\mathcal{L}_\xi^{(1)} = \bar\xi \left(i \not{D}_{\perp c} \frac{1}{i n_+ D_c} g \not{A}_{\perp \rm us} + g \not{A}_{\perp \rm us} \frac{1}{i n_+ D_c} i \not{D}_{\perp c} + [(x_\perp \partial)(g n_- A_{\rm us})] \right) \frac{\not{n}_+}{2} \xi$$

$$\begin{aligned}
\mathcal{L}_\xi^{(2)} = {} & \bar\xi \, g \not{A}_{\perp \rm us} \frac{1}{i n_+ D_c} g \not{A}_{\perp \rm us} \frac{\not{n}_+}{2} \xi - \bar\xi \, i \not{D}_{\perp c} \frac{1}{i n_+ D_c} g n_+ A_{\rm us} \frac{1}{i n_+ D_c} i \not{D}_{\perp c} \frac{\not{n}_+}{2} \xi \\
& + \bar\xi \left(i \not{D}_{\perp c} \frac{1}{i n_+ D_c} [(x_\perp \partial) \, g \not{A}_{\perp \rm us}] + [(x_\perp \partial) \, g \not{A}_{\perp \rm us}] \frac{1}{i n_+ D_c} i \not{D}_{\perp c} \right) \frac{\not{n}_+}{2} \xi \\
& + \bar\xi \left(\frac{1}{2} (n_- x) [(n_+ \partial)(g n_- A_{\rm us})] + \frac{1}{2} x_\perp^\mu x_\perp^\nu \, [\partial_\mu \partial_\nu (g n_- A_{\rm us})] \right) \frac{\not{n}_+}{2} \xi
\end{aligned}$$

and:

$$\mathcal{L}_{\xi q}^{(1)} = \bar{\xi}\, i\not{D}_{\perp c} W_c\, q + \bar{q}\, W_c^\dagger i\not{D}_{\perp c}\, \xi$$

$$\mathcal{L}_{\xi q}^{(2)} = \bar{\xi}\, g\not{A}_{\perp \mathrm{us}} W_c\, q + \bar{\xi}\, \frac{\not{n}_+}{2} \left(in_- D + i\not{D}_{\perp c} \frac{1}{in_+ D_c}\, i\not{D}_{\perp c} \right) W_c\, q$$
$$+ \bar{\xi}\, i\not{D}_{\perp c} W_c \left[(x_\perp \partial)\, q \right] + \text{"h.c."}$$

This looks straightforward and systematic, and indeed it is!

So what's the problem?

Endpoint-divergent convolution integrals arise in processes involving power-suppressed soft fields, such as q_{us} or $A_{us}^{\perp}$. These endpoint divergences spoil the (naive) scale separation accomplished with SCET.

↳ serious problem, which needs to be fixed!

Some references on endpoint divergences:

Ebert et al. (1812.08189); Moult, Stewart, Vita (1905.07411);
Moult, Stewart, Vita, Zhu (1910.14038);
Beneke, Broggio, Jaskiewicz, Vernazza (1912.01585);
Moult, Vita, Yan (1912.02188); Liu, MN (1912.08818);
Beneke et al. (2008.04943 & 2205.04479);
Liu, Mecaj, MN, Wang (2009.04456 & 2009.06779);
Liu, MN, Schnubel, Wang (2212.10447)

III. $h \to (b^* \bar{b}^*) \to \gamma\gamma$ as a Study Case

The $h \to \gamma\gamma$ decay amplitude is dominated by the top-quark loop. Since the $hq\bar{q}$ vertex is proportional to m_q, the contributions from lighter quarks are power suppressed (b, c: few % contributions; for $gg \to h$ the bottom-quark contribution is 9-13% to the rate).

But light-quark contributions are theoretically very interesting, since they give rise to an intricate pattern of large Sudakov double logarithms:

$$\mathcal{M}_b(h \to \gamma\gamma) = \frac{N_c \alpha_b}{\pi} \frac{y_b}{\sqrt{2}} m_b \, g_\perp^{\mu\nu} \varepsilon_\mu^*(k_1) \, \varepsilon_\nu^*(k_2) \left[\frac{L^2}{2} - 2 + \frac{C_F \alpha_s}{4\pi} \left(-\frac{L^4}{12} - L^3 + \dots \right) + \mathcal{O}(\alpha_s^2) \right]$$

$$\mathcal{M}_b^{\rm LL}(h \to \gamma\gamma) = \frac{N_c \alpha_b}{\pi} \frac{y_b}{\sqrt{2}} m_b \, g_\perp^{\mu\nu} \varepsilon_\mu^*(k_1) \, \varepsilon_\nu^*(k_2) \, \frac{L^2}{2} \sum_{n=0}^{\infty} \frac{2\Gamma(n+1)}{\Gamma(2n+3)} \left(-\frac{C_F \alpha_s}{2\pi} L^2 \right)^n$$

Kotsky, Yakovlev: hep-ph/9708485
Akhoury, Wang, Yakovlev: hep-ph/0102105

with:

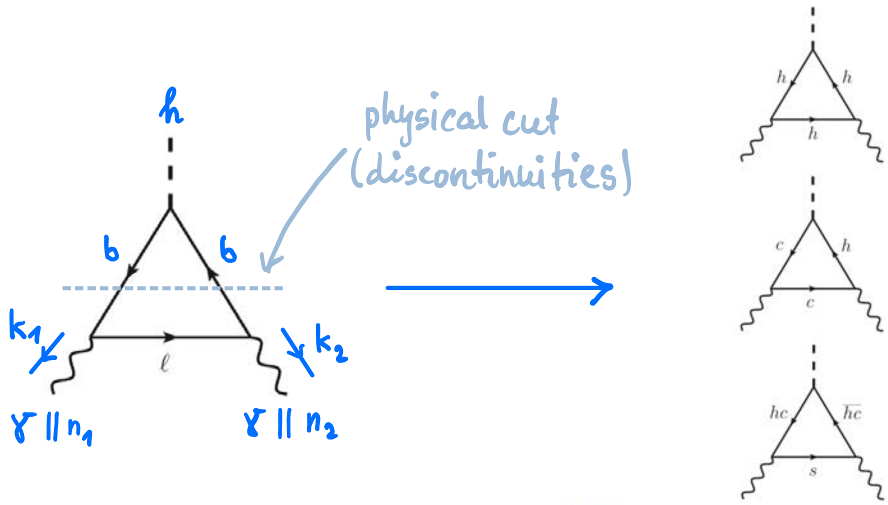
$$L = \ln \frac{-M_h^2 - i0}{m_b^2} = \ln \frac{M_h^2}{m_b^2} - i\pi$$

NLP SCET can be used to understand these logarithms beyond the leading order.

- Liu, MN (1912.08818)
- Liu, Mecaj, MN, Wang (2209.04456 & 2009.06779)

Region analysis (plus 2nd graph with opposite fermion flow):

1-loop order:



+ $(c \to \bar{c})$

hard (h) : $\ell^\mu \sim M_h\,(1,1,1)$

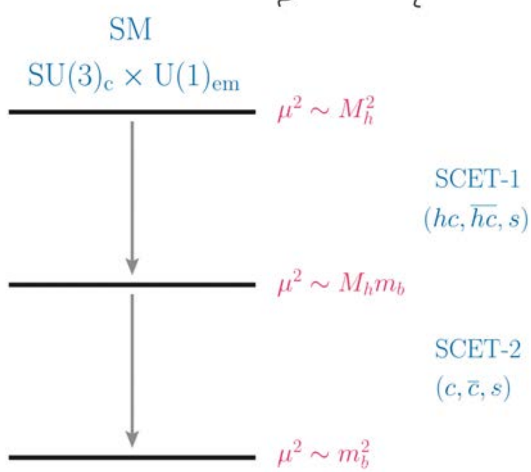
n_1-collinear (c) : $\ell^\mu \sim M_h\,(\lambda^2,1,\lambda)$

n_2-collinear $(\bar{c})$: $\ell^\mu \sim M_h\,(1,\lambda^2,\lambda)$

soft (s) : $\ell^\mu \sim M_h\,(\lambda,\lambda,\lambda)$

n_1-hard-collinear (hc) : $\ell^\mu \sim M_h\,(\lambda,1,\lambda^{\frac{1}{2}})$

n_2-hard-collinear $(\overline{hc})$: $\ell^\mu \sim M_h\,(1,\lambda,\lambda^{\frac{1}{2}})$



$$\boxed{\ell^\mu = (n_1\cdot\ell)\,\frac{n_2^\mu}{2} + (n_2\cdot\ell)\,\frac{n_1^\mu}{2} + \ell_\perp^\mu}$$

$$\boxed{\lambda = m_b/M_h \ll 1}$$

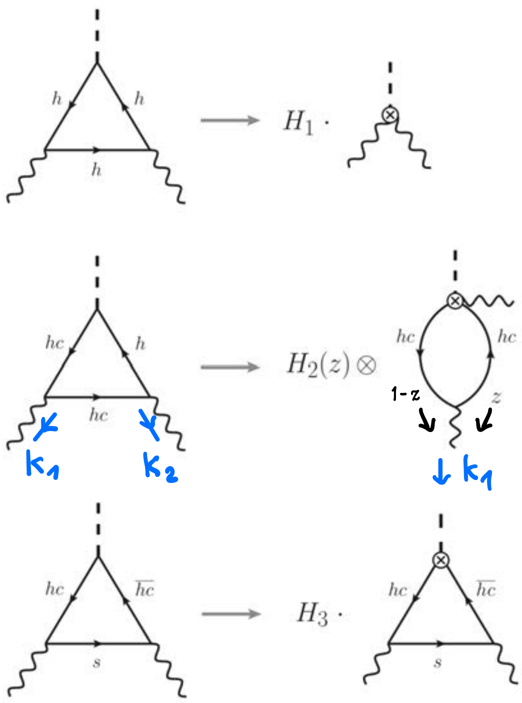
2-loop order:

(never trust a factorization theorem "derived" at 1-loop order!)

$$(\ell,p) \sim (h,h),\ (c,h),\ (c,c),\ (\bar{c},h),\ (\bar{c},\bar{c}),\ (s,h),\ (s,hc),\ (s,\overline{hc}),\ (s,s)$$

$\hookrightarrow$ same momentum modes $(\ell \leftrightarrow p)$

Matching to SCET-1 (integrate out hard modes):



SCET-1 operators

$$O_1 = \frac{m_b}{e_b^2}\, h(0)\, \mathcal{A}_{n_1}^{\perp\mu}(0)\, \mathcal{A}^{\perp}_{n_2,\mu}(0)$$

$$O_{2,n_1}(t) = h(0)\, \bar{\mathcal{X}}_{n_1}(0)\, \gamma_\perp^\mu \frac{\bar{\not{n}}_1}{2}\, \mathcal{X}_{n_1}(t\bar{n}_1)\, \mathcal{A}^{\perp}_{n_2,\mu}(0)$$

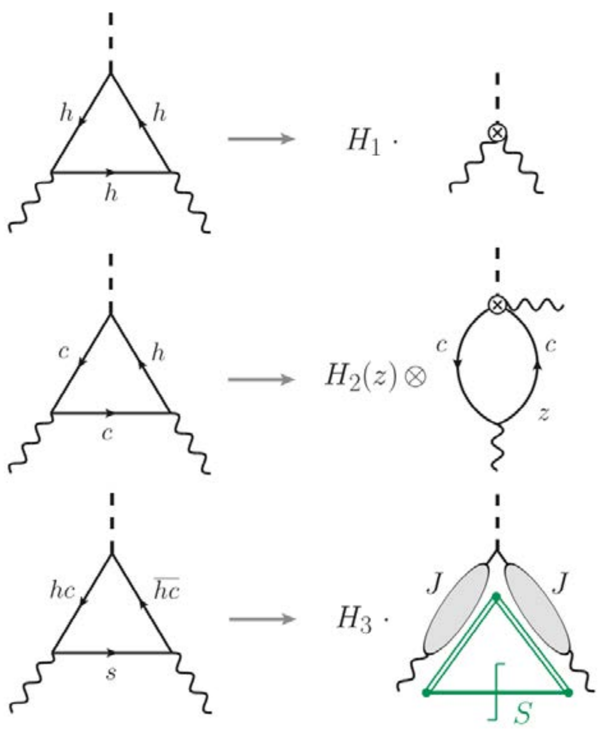
F.T.

$$O_{2,n_1}(\omega) = h(0)\left[\bar{\mathcal{X}}_{n_1}(0)\, \gamma_\perp^\mu \frac{\bar{\not{n}}_1}{2}\, \delta(\omega + i\bar{n}_1\cdot\partial)\, \mathcal{X}_{n_1}(0)\right] \mathcal{A}^{\perp}_{n_2,\mu}(0)$$

$\omega = z_1\, \bar{n}_1\cdot k_1 = z_1 M_h$

$$O_3 = T\left\{h(0)\, \bar{\mathcal{X}}_{n_1}(0)\, \mathcal{X}_{n_2}(0),\, i\int d^Dx\, \mathcal{L}^{(1/2)}_{q\,\xi_{n_1}}(x),\, i\int d^Dy\, \mathcal{L}^{(1/2)}_{\xi_{n_2} q}(y)\right\}$$

Matching to SCET-2 (integrate out hard-collinear modes):



→ lower virtuality from hc → c (else scaleless)

→ integrate out hc modes into (radiative) jet functions

Naive factorization theorem:

factor 2

$$\boxed{\mathcal{M}_b(h\to\gamma\gamma) = H_1\,\langle\gamma\gamma|\,O_1\,|h\rangle + \left[\int_0^1 dz_1\, H_2(z_1)\,\langle\gamma\gamma|\,O_2(z_1)\,|h\rangle + (z_1\to z_2)\right] + H_3\,\langle\gamma\gamma|\,O_3\,|h\rangle}$$

where:

$$\langle\gamma\gamma|\,O_3\,|h\rangle = \frac{g_\perp^{\mu\nu}}{2}\int_0^\infty \frac{d\ell_+}{\ell_+}\int_0^\infty \frac{d\ell_-}{\ell_-}\Big[J(\overbrace{\bar n_1\cdot k_1}^{M_h}\,\ell_+)\,J(\overbrace{-\bar n_2\cdot k_2}^{M_h}\,\ell_-)$$
$$+ J(-\bar n_1\cdot k_1\,\ell_+)\,J(\bar n_2\cdot k_2\,\ell_-)\Big]\,S(\ell_+\ell_-)$$

factor 2

Bare hard functions @ 2-loop order:

$$H_1 = \frac{y_{b,0}}{\sqrt2}\,\frac{N_c\alpha_{b,0}}{\pi}\,\left(-M_h^2-i0\right)^{-\epsilon}e^{\epsilon\gamma_E}\,(1-3\epsilon)\,\frac{2\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(3-2\epsilon)}$$
$$\times\left\{1-\frac{C_F\alpha_{s,0}}{4\pi}\left(-M_h^2-i0\right)^{-\epsilon}e^{\epsilon\gamma_E}\,\frac{\Gamma(1+2\epsilon)\,\Gamma^2(-2\epsilon)}{\Gamma(2-3\epsilon)}\right.$$
$$\times\left[\frac{2(1-\epsilon)(3-12\epsilon+9\epsilon^2-2\epsilon^3)}{1-3\epsilon}+\frac{8}{1-2\epsilon}\,\frac{\Gamma(1+\epsilon)\,\Gamma^2(2-\epsilon)\,\Gamma(2-3\epsilon)}{\Gamma(1+2\epsilon)\,\Gamma^3(1-2\epsilon)}\right.$$
$$\left.\left.-\frac{4(3-18\epsilon+28\epsilon^2-10\epsilon^3-4\epsilon^4)}{1-3\epsilon}\,\frac{\Gamma(2-\epsilon)}{\Gamma(1+\epsilon)\,\Gamma(2-2\epsilon)}\right]\right\}$$

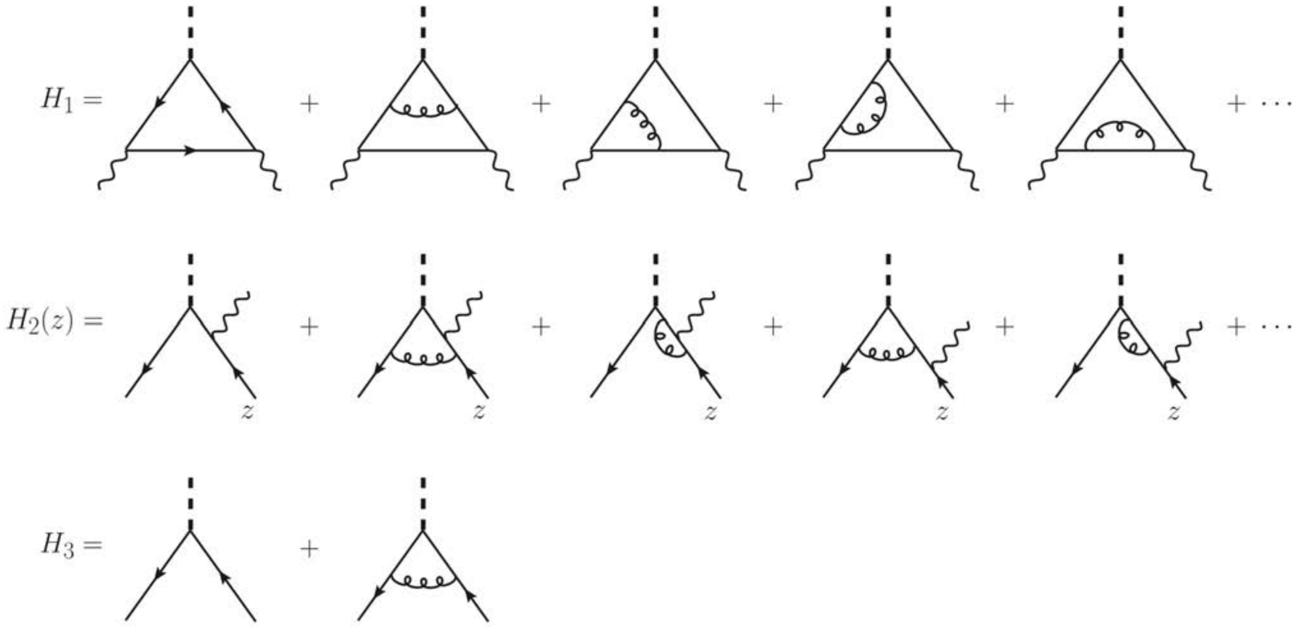
$$H_2(z) = \frac{y_{b,0}}{\sqrt2}\left\{\frac{1}{z}+\frac{C_F\alpha_{s,0}}{4\pi}\left(-M_h^2-i0\right)^{-\epsilon}e^{\epsilon\gamma_E}\,\frac{\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)}\right.$$
$$\left.\times\left[\frac{2-4\epsilon-\epsilon^2}{z^{1+\epsilon}}-\frac{2(1-\epsilon)^2}{z}-2(1-2\epsilon-\epsilon^2)\,\frac{1-z^{-\epsilon}}{1-z}\right]\right\}+(z\to1-z)$$

$$H_3 = -\frac{y_{b,0}}{\sqrt2}\left[1-\frac{C_F\alpha_{s,0}}{4\pi}\left(-M_h^2-i0\right)^{-\epsilon}e^{\epsilon\gamma_E}\,2(1-\epsilon)^2\,\frac{\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)}\right]$$

↳ dependence on hard scale $(-M_h^2-i0)$

↳ note that for $z\to0$ a second scale $(-zM_h^2-i0)$ emerges (same for $z\to1$)

Relevant diagrams:



Bare matrix elements @ 2-loop order:

O_1

$$\langle\gamma\gamma|\, O_1 \,|h\rangle = m_{b,0}\, g_\perp^{\mu\nu}$$

(valid to all orders in α_s)

O_2

$$\langle\gamma\gamma|\, O_2(z) \,|h\rangle = \frac{N_c \alpha_{b,0}}{2\pi}\, m_{b,0}\, g_\perp^{\mu\nu} \left[e^{\epsilon\gamma_E}\, \Gamma(\epsilon)\, \left(m_{b,0}^2\right)^{-\epsilon} + \frac{C_F \alpha_{s,0}}{4\pi} \left(m_{b,0}^2\right)^{-2\epsilon} \left[K(z) + K(1-z)\right] \right]$$

with:

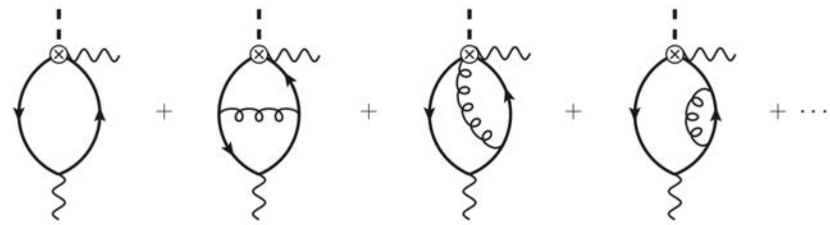
$$K(z) = \frac{1}{\epsilon^2}\left(\ln z + \frac{3}{2}\right) + \frac{1}{\epsilon}\left(\frac{\ln^2 z}{2} - \ln z \ln(1-z) - \frac{1}{4} - \frac{\pi^2}{6}\right)$$
$$+ 6\,\mathrm{Li}_3(z) + (1 - 2z - 2\ln z)\,\mathrm{Li}_2(z) + \frac{\ln^3 z}{6} + \left[z + \ln(1-z)\right] \ln^2 z$$
$$+ \left(2\,\mathrm{Li}_2(1-z) - \frac{1}{2}\ln(1-z) - \frac{1+3z}{2} - \frac{\pi^2}{6}\right) \ln z + \frac{3}{2} + \frac{\pi^2}{6} - 4\zeta_3 + \mathcal{O}(\epsilon)$$

$$\lim_{z\to 0} \left[K(z) + K(1-z)\right] = \frac{e^{2\epsilon\gamma_E}}{1-2\epsilon} \Bigg[2(2 - 3\epsilon + 2\epsilon^2)\, \Gamma^2(\epsilon) + 2(1-\epsilon)\, \Gamma(\epsilon)\, \Gamma(2\epsilon)\, \Gamma(-\epsilon)$$
$$+ z^\epsilon\, (2 - 4\epsilon - \epsilon^2)\, \frac{\Gamma(2\epsilon)\, \Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)} \Bigg]$$

↳ dependence on low scale m_b^2

↳ note that for $z \to 0$ a second scale m_b^2/z emerges (same for $z \to 1$)

Relevant diagrams for $\langle O_2 \rangle$:



Both $H_2(z)$ and $\langle O_2(z) \rangle$ are invariant under $z \leftrightarrow (1-z)$, so we can rewrite:

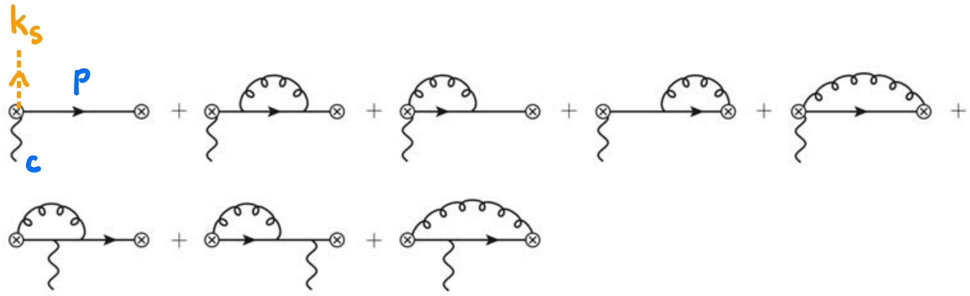
$$\int_0^1 dz\, H_2(z)\, \langle \gamma\gamma|\, O_2(z)\, |h\rangle = 2 \int_0^1 \frac{dz}{z}\, \bar{H}_2(z)\, \langle \gamma\gamma|\, O_2(z)\, |h\rangle$$

with:

$$\bar{H}_2(z) = \frac{y_{b,0}}{\sqrt{2}} \left\{ 1 + \frac{C_F \alpha_{s,0}}{4\pi} \left(-M_h^2\right)^{-\epsilon} e^{\epsilon \gamma_E} \frac{\Gamma(1+\epsilon)\, \Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)} \right.$$
$$\left. \times \left[\left(2 - 4\epsilon - \epsilon^2\right) z^{-\epsilon} - 2(1-\epsilon)^2 - 2\left(1 - 2\epsilon - \epsilon^2\right) \left[1 - (1-z)^{-\epsilon}\right] \right] \right\}$$

O_3 Bare radiative jet function at NLO:

$$J(p^2) = 1 + \frac{C_F \alpha_{s,0}}{4\pi} \left(-p^2 - i0\right)^{-\epsilon} e^{\epsilon \gamma_E} \frac{\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)} \left(2 - 4\epsilon - \epsilon^2\right)$$



Bare soft function @ 2-loop order:

$$S(\ell_+\ell_-) = -\frac{N_c \alpha_{b,0}}{\pi} m_{b,0} \left[S_a(\ell_+\ell_-)\,\theta(\ell_+\ell_- - m_{b,0}^2) + S_b(\ell_+\ell_-)\,\theta(m_{b,0}^2 - \ell_+\ell_-) \right]$$

with:

eliminated by $m_{b,0} \to m_b^{\text{pole}}$

$$S_a(w) = \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \left(w - m_{b,0}^2\right)^{-\epsilon} \qquad r = m_{b,0}^2 / w$$

$$+ \frac{C_F \alpha_{s,0}}{4\pi} \Bigg[\Big[C_1(\epsilon) + \frac{2}{\epsilon} \ln(1-r) \Big] \left(w - m_{b,0}^2\right)^{-2\epsilon} + C_2(\epsilon) \left(m_{b,0}^2\right)^{1-\epsilon} \left(w - m_{b,0}^2\right)^{-1-\epsilon}$$

$$- 2\,\mathrm{Li}_2(r) + 2 \ln r \ln(1-r) - 3 \ln^2(1-r) + 2 \ln(1-r) + \ldots \Bigg]$$

$$S_b(w) = \frac{C_F \alpha_{s,0}}{4\pi} \left(m_{b,0}^2\right)^{-2\epsilon} \left[-\frac{4}{\epsilon} \ln(1-\hat{w}) + 6 \ln^2(1-\hat{w}) + \ldots \right]$$

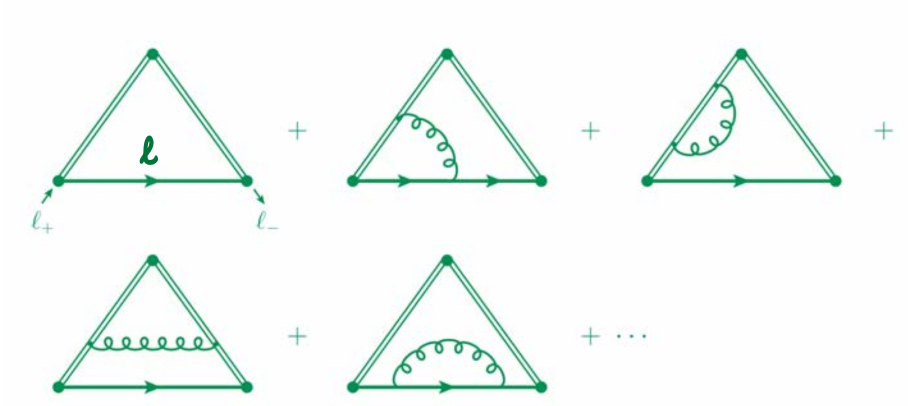
$\sim w$ for $w \to 0$

$$\hat{w} = w / m_{b,0}^2$$

and:

$$C_1(\epsilon) = \frac{2 e^{2\epsilon\gamma_E}}{\Gamma(1-2\epsilon)} \left[\frac{(1+\epsilon)\,\Gamma(-\epsilon)^2}{\Gamma(2-2\epsilon)} + 2\Gamma(\epsilon)\,\Gamma(-\epsilon) \right]$$

$$C_2(\epsilon) = -2 e^{2\epsilon\gamma_E} \frac{3-2\epsilon}{1-2\epsilon} \frac{\Gamma(\epsilon)}{\Gamma(-\epsilon)}$$



Endpoint-divergent convolution integrals (I):

Note that for $z \to 0$:

$$\bar{H}_2(z) \sim 1 + \alpha_s \left(c_0(\epsilon) + c_1(\epsilon)\, z^{-\epsilon} \right) + \dots$$

$$\langle O_2(\epsilon) \rangle \sim 1 + \alpha_s \left(d_0(\epsilon) + d_1(\epsilon)\, z^{\epsilon} \right) + \dots$$

$$\Rightarrow \int_0^1 \frac{dz}{z}\, \bar{H}_2(z) \langle O_2(\epsilon) \rangle$$

$$= \int_0^1 \frac{dz}{z} \left\{ 1 + \alpha_s (c_0 + d_0) + \alpha_s^2 (c_0 d_0 + c_1 d_1) + \dots \right\} \quad \text{unregularized}$$

$$+ \int_0^1 dz \left\{ \alpha_s c_1 \frac{1}{z^{1+\epsilon}} + \alpha_s d_1 \frac{1}{z^{1-\epsilon}} + \dots \right\} \quad \text{dimensionally regularized}$$

Note also that we cannot use the standard formula

$$\frac{1}{z^{1-\epsilon}} = \frac{1}{\epsilon}\, \delta(z) + \left(\frac{1}{z} \right)_+ + \dots$$

in $H_2(z)$, since $\langle O_2(z) \rangle$ itself is singular at $z = 0$!

Endpoint-divergent convolution integrals (II):

Considering the double convolution

$$\langle\gamma\gamma|\,O_3\,|h\rangle = \frac{g_\perp^{\mu\nu}}{2}\int_0^\infty \frac{d\ell_+}{\ell_+}\int_0^\infty \frac{d\ell_-}{\ell_-}\Big[J(\bar n_1\cdot k_1\,\ell_+)\,J(-\bar n_2\cdot k_2\,\ell_-) + J(-\bar n_1\cdot k_1\,\ell_+)\,J(\bar n_2\cdot k_2\,\ell_-)\Big]\,S(\ell_+\ell_-)$$

we first observe that there are no singularities for $\ell_\pm\to 0$, since the soft function $S(\ell_+\ell_-)\propto \ell_+\ell_-$ in this region (see p. 13). There are, however, endpoint singularities for $\ell_\pm\to\infty$, since:

$$\left.\begin{aligned} J(p^2) &= 1+\alpha_s\, c(\epsilon)\,(-p^2-i0)^{-\epsilon}+\dots \\ S(w) &\overset{w\to\infty}{=} w^{-\epsilon}+\alpha_s\, d(\epsilon)\, w^{-2\epsilon}+\dots \end{aligned}\right\}\ \text{leads to log divergences}$$

At first sight, it looks like these divergences are regularized dimensionally (but still intertwined between soft and hard-collinear dynamics). Yet, this is not the case, since at fixed $w=\ell_+\ell_-$ the integration along the hyperbola $w=$ const. generates a rapidity divergence.

To show this, we rewrite:

$$\int_0^\infty\frac{d\ell_-}{\ell_-}\int_0^\infty\frac{d\ell_+}{\ell_+} = \int_0^\infty\frac{d\ell_-}{\ell_-}\int_0^{\ell_-}\frac{d\ell_+}{\ell_+} + \int_0^\infty\frac{d\ell_+}{\ell_+}\int_0^{\ell_+}\frac{d\ell_-}{\ell_-}$$

$$= \int_0^\infty\frac{d\ell_-}{\ell_-}\int_0^{\ell_-^2}\frac{dw}{w} + \int_0^\infty\frac{d\ell_+}{\ell_+}\int_0^{\ell_+^2}\frac{dw}{w} = \int_0^\infty\frac{dw}{w}\left(\int_{\sqrt{w}}^\infty\frac{d\ell_-}{\ell_-} + \int_{\sqrt{w}}^\infty\frac{d\ell_+}{\ell_+}\right)$$

This leads to:

$$\langle O_3\rangle \sim \underbrace{\int_0^\infty \frac{dw}{w}\, S(w)}_{\text{dimensionally regularized}} \left(\int_{\sqrt{w}}^\infty \frac{d\ell_-}{\ell_-} + \int_{\sqrt{w}}^\infty \frac{d\ell_+}{\ell_+}\right) \underbrace{J(M_h\ell_+)\, J(-M_h\ell_-)}_{1+\alpha_s c(\epsilon)\left[(-M_h\ell_+ - i0)^{-\epsilon} + (M_h\ell_-)^{-\epsilon}\right] + \dots}$$

↪ unregularized divergences

Introducing a rapidity regulator:

None of the regulators available in late 2019 worked for us, since they would break the analytic structure of the amplitude. Instead we used:

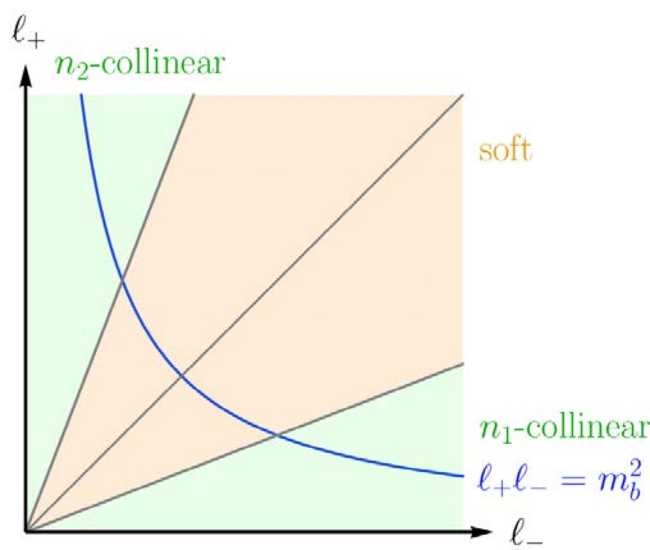
$$\left(\frac{-z_i M_h^2 - i0}{\nu^2}\right)^\eta ; \quad i=1,2$$

for $\int H_2 \langle O_2\rangle$

$$\left(\frac{\mp \overbrace{\bar{n}_1\cdot k_1}^{M_h}\,\ell_+ \pm \overbrace{\bar{n}_2\cdot k_2}^{M_h}\,\ell_- - i0}{\nu^2}\right)^\eta$$

for $\langle O_3\rangle \sim \iint JJS$

$$\left(\frac{\mp 2M_h\ell_3 - i0}{\nu^2}\right)^\eta$$



If we identify $z_1 = \ell_-/\bar{n}_1\cdot k_1$ and $z_2 = \ell_+/\bar{n}_2\cdot k_2$, we see that the regulators match up at the boundaries between the soft and hard-collinear regions (recall that $M_h^2 = \bar{n}_1\cdot k_1\, \bar{n}_2\cdot k_2$).

We then obtain the regularized (naive) factorization theorem:

$$\begin{aligned}\mathcal{M}_b(h\to\gamma\gamma) &= \lim_{\eta\to 0}\, H_1\,\langle\gamma\gamma|\,O_1\,|h\rangle \\ &\quad + 2\int_0^1 \frac{dz_1}{z_1}\left(\frac{-z_1M_h^2 - i0}{\nu^2}\right)^{\eta} \bar{H}_2(z_1)\,\langle\gamma\gamma|\,O_2(z_1)\,|h\rangle + (z_1\to z_2) \\ &\quad + \frac{g_\perp^{\mu\nu}}{2}\,H_3\int_0^\infty\frac{d\ell_-}{\ell_-}\int_0^\infty\frac{d\ell_+}{\ell_+}\,S(\ell_+\ell_-) \\ &\quad \times\left[\left(\frac{-\bar{n}_1\cdot k_1\,\ell_+ + \bar{n}_2\cdot k_2\,\ell_- - i0}{\nu^2}\right)^{\eta} J(\bar{n}_1\cdot k_1\,\ell_+)\,J(-\bar{n}_2\cdot k_2\,\ell_-)\right. \\ &\qquad \left. + \left(\frac{\bar{n}_1\cdot k_1\,\ell_+ - \bar{n}_2\cdot k_2\,\ell_- - i0}{\nu^2}\right)^{\eta} J(-\bar{n}_1\cdot k_1\,\ell_+)\,J(\bar{n}_2\cdot k_2\,\ell_-)\right]\end{aligned}$$

Explicit evaluation of the three terms (in units of the LO amplitude yields $\mathcal{M}_b(h\to\gamma\gamma) = \mathcal{M}_0\,(T_1+T_2+T_3)$, with:

polynomials

$$T_1 = \frac{1}{\epsilon^2} - \frac{L_h}{\epsilon} + \frac{L_h^2}{2} - 2 - \frac{\pi^2}{12} + \frac{C_F\alpha_s}{4\pi}\,k_1(L_h)$$

$$T_2 = \left[\frac{2}{\eta} + 2\ln\frac{-M_h^2 - i0}{\nu^2}\right]\left[\frac{1}{\epsilon} - L_m + \frac{C_F\alpha_s}{4\pi}\,k_0(L_h,L_m)\right] + \frac{C_F\alpha_s}{4\pi}\,k_2(L_h,L_m)$$

$$\begin{aligned}T_3 &= -\left[\frac{2}{\eta} + \ln\frac{-M_h^2 - i0}{\nu^2} + \ln\frac{m_b^2}{\nu^2}\right]\left[\frac{1}{\epsilon} - L_m + \frac{C_F\alpha_s}{4\pi}\,k_0(L_h,L_m)\right] \\ &\quad - \frac{1}{\epsilon^2} + \frac{L_m}{\epsilon} - \frac{L_m^2}{2} + \frac{\pi^2}{12} + \frac{C_F\alpha_s}{4\pi}\,k_3(L_h,L_m)\end{aligned}$$

and:

$$\mathcal{M}_0 = \frac{N_c\alpha_b}{\pi}\,\frac{y_b(\mu)}{\sqrt{2}}\,m_b(\mu)\,g_\perp^{\mu\nu}\,\varepsilon_\mu^*(k_1)\,\varepsilon_\nu^*(k_2)$$

We see that the dependence on the rapidity regulator η (and the associated scale ν) cancel in the sum $T_2 + T_3$, leaving behind a large logarithm $\ln(-M_h^2/m_b^2)$.

IV. Systematic Removal of Endpoint Divergences

The fact that rapidity divergences cancel between T_2 and T_3 looks like a miracle, given that T_2 contains hard and collinear functions, while T_3 contains hard, hard-collinear and soft objects. How can this possibly work to all orders in perturbation theory?

Besides the rapidity divergences, T_2 and T_3 are still plagued by (dimensionally regularized) endpoint-divergent convolution integrals, which give rise to $1/\epsilon^n$ pole that cannot be subtracted by renormalizing the individual component functions in the (naive) factorization theorem.

↳ scale separation is broken!

↳ endpoint divergences must be removed systematically to all orders in perturbation theory

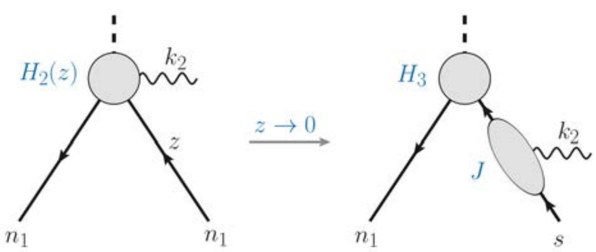
It turns out that both problems are solved by the same mechanism!

D-dimensional refactorization conditions:

The all-order cancellation of rapidity divergences between T_2 and T_3 is possible only if the (bare) integrands of the two terms are related to each other in the singular limits. Specifically, the D-dimensional relations:

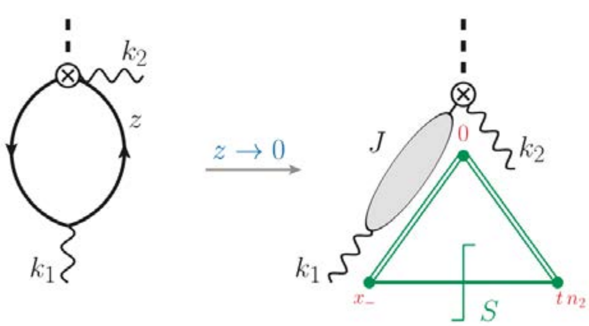
$$\boxed{\begin{aligned} &[\![\bar{H}_2(z)]\!] = -H_3\, J(zM_h^2) \\ &[\![\langle\gamma\gamma|\, O_2(z)\, |h\rangle]\!] = -\frac{g_\perp^{\mu\nu}}{2}\int_0^\infty \frac{dw}{w}\, J\Big(-\frac{w}{z}\Big)\, S(w) = -\frac{g_\perp^{\mu\nu}}{2}\int_0^\infty \frac{d\ell_+}{\ell_+}\, J(-\overbrace{\bar{n}_1\cdot k_1\, \ell_+}^{\text{used } z = \ell_-/\bar{n}_1\cdot k_1})\, S(\ell_+\ell_-) \end{aligned}}$$

must hold to all orders of perturbation theory, where $[\![f(z)]\!]$ means the leading terms of the function $f(z)$ for $z \to 0$. Using the results on p. 10-11, one finds that these relations are indeed satisfied at 2-loop order. Using SCET tools, we have proved them to all orders.



Scales in H_2 for $z \sim \lambda$:

M_h^2 (h) and $z M_h^2 \sim \lambda$ (hc) (cf p.10)



Scales in O_2 for $z \sim \lambda$:

$m_b^2 \sim \lambda^2$ (s) and $\frac{m_b^2}{z} \sim \lambda$ (hc)

(cf p.12)

To prove that these condition guarantee the cancellation of rapidity divergences, we make the general ansatz for the bare jet function:

$$J(p^2) = \sum_{n=0}^{\infty} c_n(\epsilon) \left(\frac{\alpha_{s,0}}{4\pi}\right)^n \left(-p^2 - i0\right)^{-n\epsilon}$$

and evaluate the second and third term of the factorization theorem on p.17:

$$\begin{aligned} 4\bar{H}_2 \otimes \langle O_2\rangle &= 4\,[\![\bar{H}_2]\!] \otimes [\![\langle O_2\rangle]\!] \quad + \text{ regular terms} \\ &= 2g_\perp^{\mu\nu} H_3 \int_0^\infty \frac{dw}{w}\, S(w) \int_0^{M_h} \frac{d\ell_-}{\ell_-} \left(\frac{-M_h\ell_- - i0}{\nu^2}\right)^\eta J(-M_h w/\ell_-)\, J(M_h\ell_-) + \ldots \\ &= 2g_\perp^{\mu\nu} H_3 \int_0^\infty \frac{dw}{w}\, S(w) \Bigg\{ \left[\frac{1}{\eta} + \ln\frac{-M_h^2 - i0}{\nu^2} + \mathcal{O}(\eta)\right] \\ &\qquad \times \sum_{n=0}^{\infty} c_n^2(\epsilon) \left(\frac{\alpha_{s,0}}{4\pi}\right)^{2n} \left(-wM_h^2 - i0\right)^{-n\epsilon} \\ &\qquad - \sum_{m\neq n} \frac{c_m(\epsilon)\, c_n(\epsilon)}{(m-n)\,\epsilon} \left(\frac{\alpha_{s,0}}{4\pi}\right)^{m+n} \left(-M_h^2 - i0\right)^{-m\epsilon} w^{-n\epsilon} \Bigg\} \end{aligned}$$

all-order cancellation

and:

$$\begin{aligned} H_3\,\langle O_3\rangle = -g_\perp^{\mu\nu} H_3 \int_0^\infty \frac{dw}{w}\, S(w) \left[\frac{2}{\eta} + \ln\frac{-M_h^2 - i0}{\nu^2} + \ln\frac{w}{\nu^2} + \mathcal{O}(\eta)\right] \\ \times \sum_{n=0}^{\infty} c_n^2(\epsilon) \left(\frac{\alpha_{s,0}}{4\pi}\right)^{2n} \left(-wM_h^2 - i0\right)^{-n\epsilon} \end{aligned}$$

The remaining terms in the sum:

$$\lim_{\eta\to 0} 4\,[\![\bar{H}_2]\!] \otimes [\![\langle O_2\rangle]\!] + H_3\,\langle O_3\rangle$$
$$= g_\perp^{\mu\nu} H_3 \int_0^\infty \frac{dw}{w}\, S(w) \left[\ln\frac{-M_h^2 - i0}{w} \sum_{n=0}^{\infty} c_n^2(\epsilon) \left(\frac{\alpha_{s,0}}{4\pi}\right)^{2n} \left(-wM_h^2 - i0\right)^{-n\epsilon} \right.$$
$$\left. - 2 \sum_{m\neq n} \frac{c_m(\epsilon)\, c_n(\epsilon)}{(m-n)\,\epsilon} \left(\frac{\alpha_{s,0}}{4\pi}\right)^{m+n} \left(-M_h^2 - i0\right)^{-m\epsilon} w^{-n\epsilon} \right]$$

are free of rapidity divergences.

Side remark:

It is instructive to compare this result with the simpler case of massive Sudakov resummation at leading power studied e.g. by Chiu, Golf, Kelley and Manohar in 0709.2377 and 0712.0396. In this case (e.g. $Z' \to b\bar{b}$) there is no analogue of O_1. More importantly, there are no jet functions, because soft boson exchanges between (hard-) colliner quarks are described by soft Wilson lines. Hence $c_n(\epsilon) = 0$ for all $n \neq 0$, which leads to drastic simplifications:

$$4\,[\![\bar{H}_2]\!] \otimes [\![\langle O_2\rangle]\!] + H_3\,\langle O_3\rangle \to g_\perp^{\mu\nu} H_3 \left[\ln\frac{-M_h^2 - i0}{m_{b,0}^2} \int_0^\infty \frac{dw}{w}\, S(w) - \int_0^\infty \frac{dw}{w}\, S(w)\, \ln\frac{w}{m_{b,0}^2} \right]$$

two soft matrix elements

linear function of rapidity logarithm!

Subtraction of endpoint divergences:

We can use the exact refactorization conditions from p. 20:

$$[\![\bar{H}_2(z)]\!] = -H_3\, J(zM_h^2)$$

$$[\![\langle\gamma\gamma|\, O_2(z)\,|h\rangle]\!] = -\frac{g_\perp^{\mu\nu}}{2}\int_0^\infty \frac{dw}{w}\, J\Big(-\frac{w}{z}\Big)\, S(w) = -\frac{g_\perp^{\mu\nu}}{2}\int_0^\infty \frac{d\ell_+}{\ell_+}\, J(-\bar{n}_1\cdot k_1\,\ell_+)\, S(\ell_+\ell_-)$$

to remove the endpoint divergences in the naive factorization formula. We find:

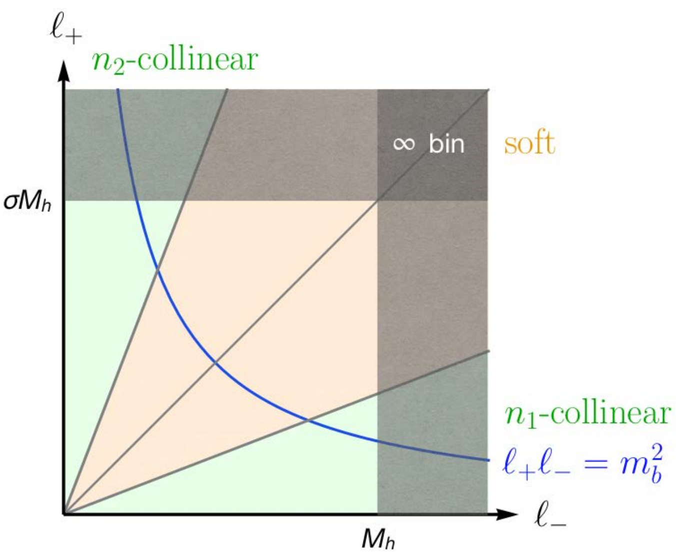
$$\begin{aligned}\mathcal{M}_b(h\to\gamma\gamma) &= (H_1+\Delta H_1)\,\langle\gamma\gamma|\, O_1\,|h\rangle \\ &+ 4\lim_{\delta\to 0}\int_\delta^1 \frac{dz}{z}\Big[\bar{H}_2(z)\,\langle\gamma\gamma|\, O_2(z)\,|h\rangle - [\![\bar{H}_2(z)]\!]\,[\![\langle\gamma\gamma|\, O_2(z)\,|h\rangle]\!]\Big] \\ &+ g_\perp^{\mu\nu} H_3 \lim_{\sigma\to -1}\int_0^{M_h}\frac{d\ell_-}{\ell_-}\int_0^{\sigma M_h}\frac{d\ell_+}{\ell_+}\, J(M_h\ell_-)\, J(-M_h\ell_+)\, S(\ell_+\ell_-)\Big|_{\text{leading power}}\end{aligned} \qquad (*)$$

power corrections $\sim \frac{m_b^2}{M_h^2}$ should be dropped

where:

$$\Delta H_1 = -\lim_{\sigma\to -1} H_3 \int_{M_h}^\infty \frac{d\ell_-}{\ell_-}\int_{\sigma M_h}^\infty \frac{d\ell_+}{\ell_+}\, J(M_h\ell_-)\, J(-M_h\ell_+)\, \frac{S_\infty(\ell_+\ell_-)}{m_b}$$

"∞ bin subtraction"



↳ $S_\infty(w) \equiv [\![S(w)]\!]$ denotes the leading terms in the limit $w \gg m_b^2$

In the rearranged factorization theorem, all endpoint divergences have been removed by means of suitable subtractions!

↳ no need for analytic regulators

To show that this is a consistent factorization theorem, one must show that the form (*) persists after renormalization. We have shown that this is indeed the case!

see: Liu, Mecaj, MN, Wang (2009.06779)

Using the RG equations for the hard, jet and soft functions and for the operators $O_i(\mu)$, we have predicted the first four logarithms:

$$\sim \alpha \alpha_s^2 c_n L^n \quad \text{with} \quad n = 6, 5, 4, 3$$

in the 3-loop decay amplitude, finding agreement with the fixed-order calculation in Harlander, Prausa, Usovitsch (1907.06957).

By solving the RG equations at subleading logarithmic order, we have resummed the infinite towers of large logarithms of the form:

$$\alpha L^2 \left\{ (\alpha_s L^2)^n, \ \alpha_s L (\alpha_s L^2)^n \right\}$$

to all orders of perturbation theory, correcting an error in the subleading term.

see: Liu, Mecaj, MN, Wang (2009.04456)

V. General Lessons

Refactorization-based subtraction (RBS) scheme:

- endpoint divergences are spurious (overlap of soft and collinear regions) and must cancel out in physical results

- exact D-dimensional refactorization relations relate the (bare) integrands of different terms in the naive factorization theorem in the singular limits

- allows for systematic removal of endpoint divergences by reshuffling (our example: $H_2 \otimes \langle O_2 \rangle$ mixes into $H_1 \langle O_1 \rangle$ and $H_3 \langle O_3 \rangle = H_3\, J \otimes J \otimes S$)

- refactorized structure survives after renormalization (it has to!)

- example studied here is not the simplest one: often there is just one collinear direction, and endpoint divergences result from the overlap of collinear and soft regions

- RBS scheme works also in a nonperturbative context, where $\langle O_i \rangle$ are nonperturbative hadronic objects;

  see e.g.: Feldmann, Gubernari, Huber, Seitz (2211.04209)

  Cornella, Ferré, König, MN (2212.14430 & 2601.14361)