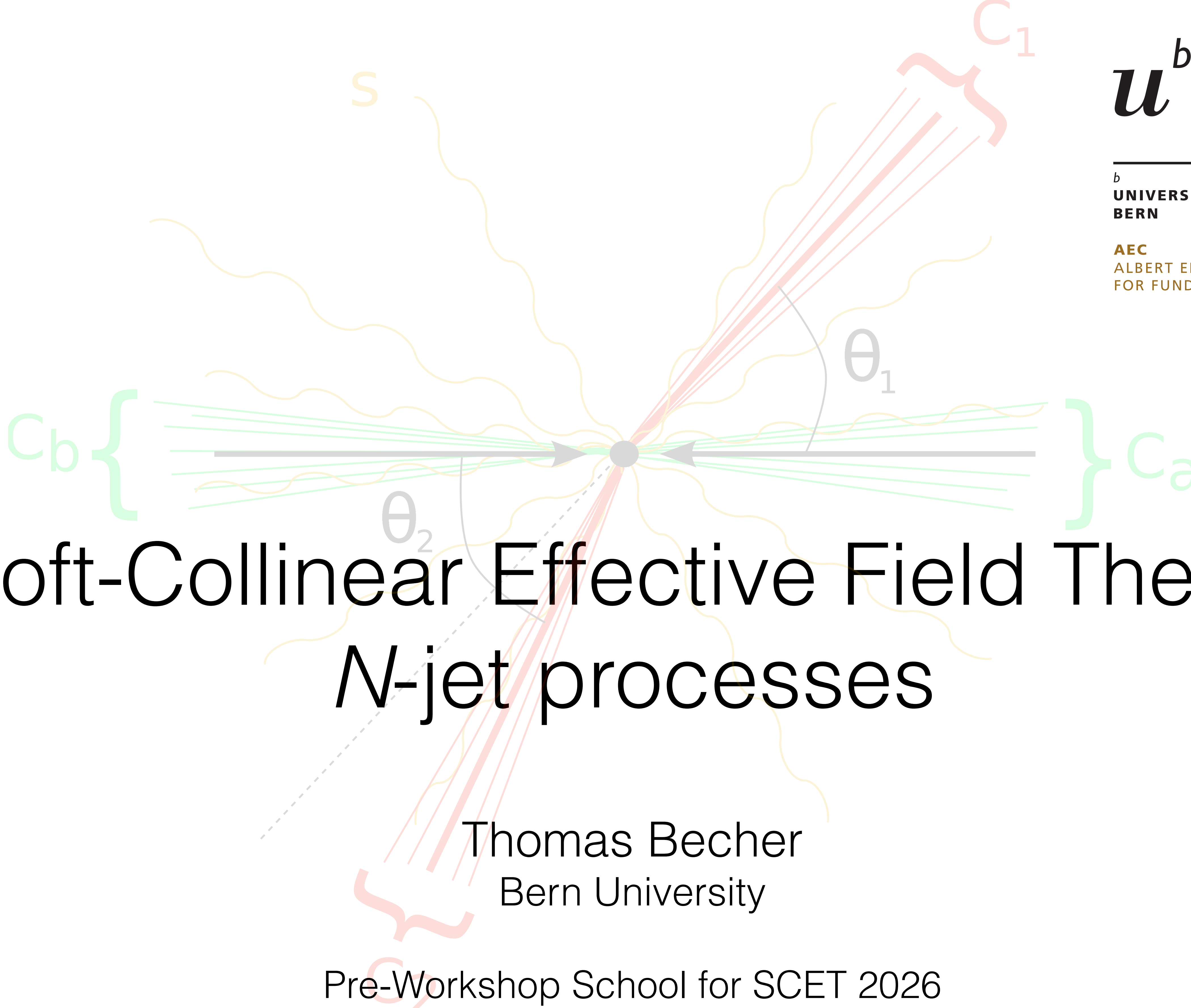



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# Soft-Collinear Effective Field Theory
# *N*-jet processes


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Pre-Workshop School for SCET 2026
KIAS Korea, Feb. 26-28, 2026

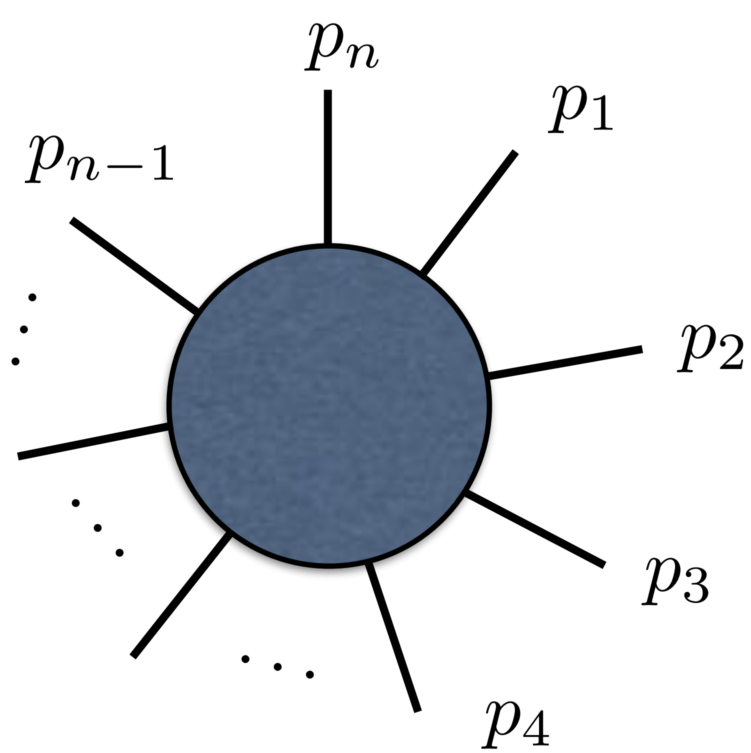
# $N$-jet processes

The simplest application of SCET involve energetic particles in two directions. Interesting to look at the generalization to more directions for

- jet processes at LHC (three directions, even for 1-jet processes!)
- non-global observables (sensitive to emissions from individual hard partons)
- study IR structure of $N$-point amplitudes

# $n$-point Green's function

Start with $n$-point off-shell Green's functions in the limit of small off-shellness $p_i^2$ and large momentum transfers $s_{ij}$.




$$s_{ij} = (p_i \pm p_j)^2$$

+ if both incoming or outgoing
– otherwise

$$p_k^2 \ll |s_{ij}|$$

Introduce reference vectors $n_i = (1, \hat{\boldsymbol{n}}_i)$ and $\bar{n}_i = (1, -\hat{\boldsymbol{n}}_i)$ with $n_i \cdot \bar{n}_i = 2$ .

# SCET for $n$-jet processes

The effective theory contains collinear fields $\xi_i$ , $A_i^\mu$ for each direction and a soft fields $q_s$, $A_s^\mu$ which interact with the collinear sectors.

Convenient to use building blocks

$$\chi_i(x) = W_i^\dagger(x)\,\xi_i(x) = W_i^\dagger(x)\,\frac{\not{n}_i\bar{\not{n}}_i}{4}\,\psi_i(x)\,,$$

$$\mathcal{A}_\perp^\mu(x) = W_i^\dagger(x)\,[iD_\perp^\mu, W_i(x)]\,.$$

where $$W_i(x) = \mathbf{P}\,\exp\left(ig\int_{-\infty}^0 ds\,\bar{n}_i\cdot A_i(x+s\bar{n}_i)\right)$$

Note: the $\mathcal{A}_{i\perp}^\mu$ components are sufficient since $n\cdot\mathcal{A}_i$ is power suppressed and $\bar{n}\cdot\mathcal{A}_i = 0$ .

# Collinear Lagrangian

The collinear Lagrangians have exactly the same in each sector, only the reference vectors are different.

$$\mathcal{L}_{\rm SCET} = \bar{q}_s\, i\not{D}_s\, q_s - \frac{1}{4}(F^{s\,a}_{\mu\nu})^2$$

$$+ \sum_{i=1}^{n} \left\{ \bar{\xi}_i \,\frac{\not{\bar{n}}}{2} \left[ i n_i \cdot D + i\not{D}_{ci\perp} \frac{1}{i\bar{n}_i \cdot D_{ci}} i\not{D}_{ci\perp} \right] \xi_i - \frac{1}{4}(F^{ci\,a}_{\mu\nu})^2 \right\}$$

Only the $n_i \cdot A_s(x_-)$ components interacts with the $i$-collinear fields.

$$\left[\ x_-^\mu = \bar{n}_i \cdot x \,\frac{n_i^\mu}{2}\ \right]$$

# Decoupling transformation

The soft interactions can be decouped from the collinear Lagrangian by the field redefinitions

$$\chi_i(x) = S_i(x_-)\, \chi_i^{(0)}(x)\,,$$
$$\mathcal{A}_{i\perp}^{\mu}(x) = S_i(x_-)\, \mathcal{A}_{i\perp}^{(0)\mu}(x)\, S_i^\dagger(x_-)$$

with

$$S_i(x) = \mathbf{P}\exp\left(ig\int_{-\infty}^{0} dt\, n_i\cdot A_s^a(x+tn_i)\, t^a\right),$$

To have a unified treatment of quarks and gluons, let's denote a generic collinear field by $(\phi_i)^{\alpha_i}_{a_i}(x)$ , where $a_i$ is the color index and $\alpha_i$ the Lorentz/Dirac index. For such a field, the decoupling takes the form

$$(\phi_i)^{\alpha_i}_{a_i}(x) = [\boldsymbol{S}_i(x_-)]_{a_i b_i}\, (\phi_i)^{(0)\,\alpha_i}_{\quad b_i}(x)$$

where $\boldsymbol{S}_i$ is the soft Wilson line in the appropriate color representation $\boldsymbol{T}_i^a$. For example

$$\boldsymbol{T}_i^a = t^a \qquad \text{for quarks}$$

$$\boldsymbol{T}_i^a = -(t^a)^T \qquad \text{for anti-quark}$$

gives anti-path ordering

# Color-space formalism

Basetto, Ciafaloni, Marchesini ’84; Catani, Seymour ‘96

Represent amplitudes as vectors in color space:

$$\left|c_1, c_2, \ldots, c_n\right\rangle$$

color index of first parton

Color generator for i[th] parton $\boldsymbol{T}_i^a \left|c_1, c_2, \ldots, c_n\right\rangle$ acts as a matrix:

- $t^a$ for quarks, $-(t^a)^T$ for q’s and $-i f^{abc}$ for gluons
- product $\boldsymbol{T}_i \cdot \boldsymbol{T}_j = \sum_a \boldsymbol{T}_i^a \boldsymbol{T}_j^a$ (commutative)
- charge conservation $\sum_i \boldsymbol{T}_i^a = 0$ implies:

$$\sum_{(i,j)} \boldsymbol{T}_i \cdot \boldsymbol{T}_j = -\sum_i \boldsymbol{T}_i^2 = -\sum_i C_i$$

i≠j

$C_F$ or $C_A$

# Leading-power $n$-jet operators

The leading power operator contains *exactly one* collinear field ( $\chi_i$ , $\bar{\chi}_i$ , or $\mathcal{A}_{i\perp}$) from each sector.

$$\mathcal{H}_n^{\text{eff}} = \int dt_1 \, \ldots dt_n \, \tilde{\mathcal{C}}^{a_1 \ldots a_n}_{\alpha_1 \ldots \alpha_n}(t_1, \ldots, t_n, \mu)$$

$$\times \, (\phi_1)^{\alpha_1}_{a_1}(x + t_1 \bar{n}_1) \ldots (\phi_n)^{\alpha_n}_{a_n}(x + t_n \bar{n}_n)$$

Uncommon to let the Wilson coefficients depend on color and spin indices, but it allow us to write $\mathcal{H}_n^{\text{eff}}$ in color-space notation as

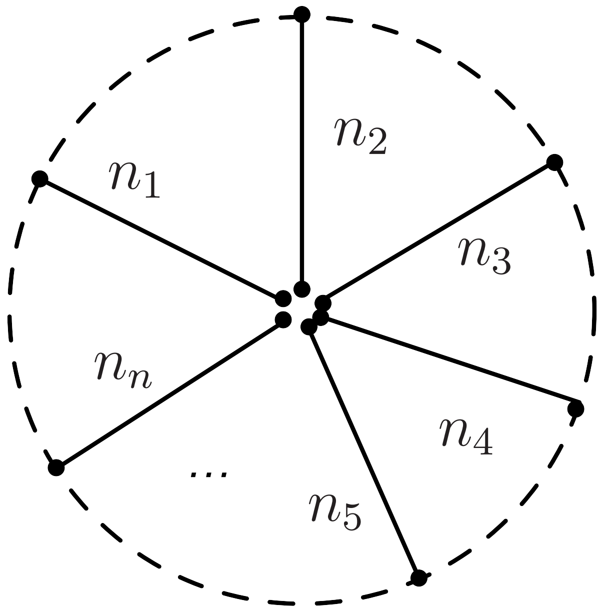
$$\mathcal{H}_n^{\text{eff}} = \int dt_1 \, \ldots \, dt_n \, \langle O_n(\{\underline{t}\}, \mu) | \tilde{\mathcal{C}}_n(\{\underline{t}\}, \mu) \rangle$$

# Soft function

The SCET decoupling transformation removes soft interactions among collinear fields and absorbs them into soft Wilson lines

$n_i \sim p_i$ light-like reference vector

$$\boldsymbol{S}_i = \mathbf{P} \exp \left[ ig \int_{-\infty}^{0} dt\, n_i \cdot A_a(tn_i)\, T_i^a \right]$$

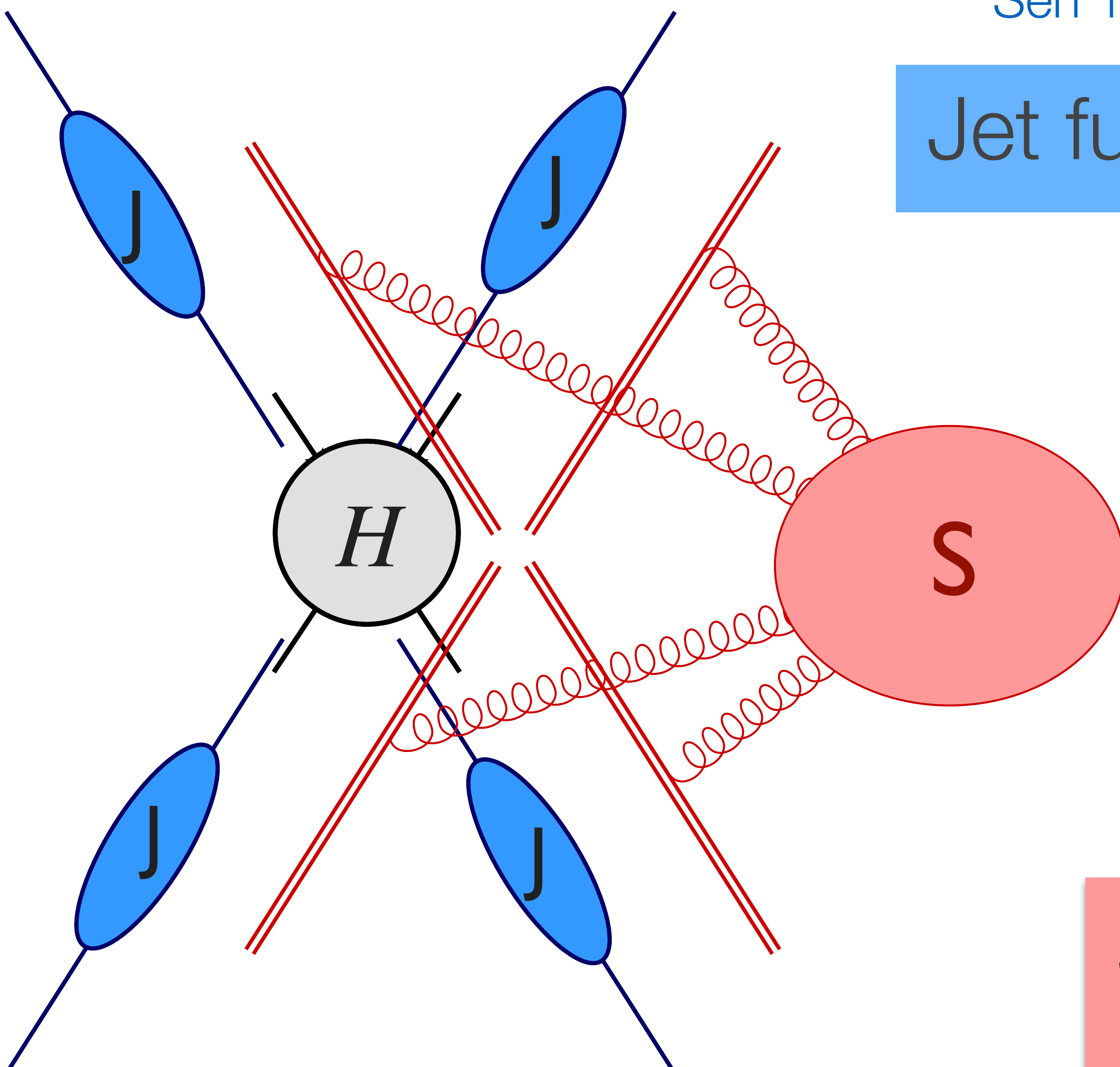



For n-jet operator one gets:

$$\mathcal{S}(\{\underline{n}\}, \mu) = \langle 0 | \boldsymbol{S}_1(0) \dots \boldsymbol{S}_n(0) | 0 \rangle = \exp(\tilde{\mathcal{S}}(\{\underline{n}\}, \mu))$$

# Soft-collinear factorization: *n* jet case

Sen 1983; Kidonakis, Oderda, Sterman 1998




Jet functions $J_i = J_i\,(p_i^2)$

Hard function $H \equiv C_n$ depends on large momentum transfers $s_{ij}$ between jets

Soft function *S* depends on scales $\Lambda_{ij} = \dfrac{p_i^2 p_j^2}{s_{ij}}$

# One-loop result

An explicit calculation gives the the following results for the divergent part of the soft and jet functions

$$\mathcal{J}_q(p^2,\mu) = 1 + \frac{\alpha_s}{4\pi}\,C_F \left( \frac{2}{\epsilon^2} + \frac{2}{\epsilon}\,\ln\frac{\mu^2}{-p^2} + \frac{3}{2\epsilon} \right) + \mathcal{O}(\epsilon^0)\,,$$

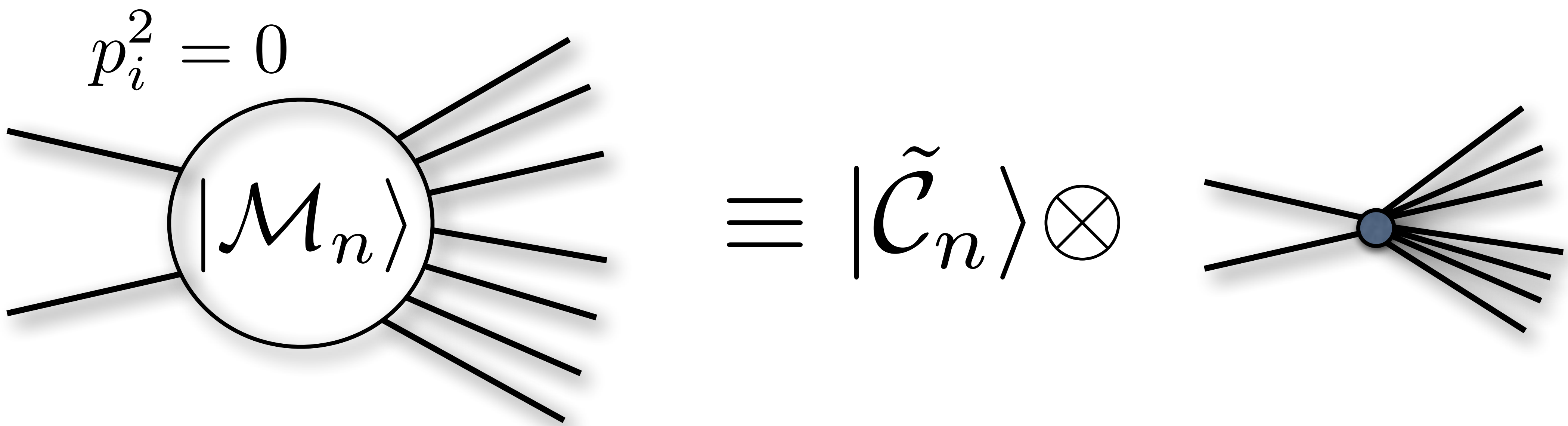
$$\mathcal{J}_g(p^2,\mu) = 1 + \frac{\alpha_s}{4\pi} \left[ C_A \left( \frac{2}{\epsilon^2} + \frac{2}{\epsilon}\,\ln\frac{\mu^2}{-p^2} \right) + \frac{\beta_0}{2\epsilon} \right] + \mathcal{O}(\epsilon^0)\,,$$

$$\mathcal{S}(\{\underline{p}\},\mu) = 1 + \frac{\alpha_s}{4\pi} \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2} \left( \frac{2}{\epsilon^2} + \frac{2}{\epsilon}\,\ln\frac{-s_{ij}\,\mu^2}{2(-p_i^2)(-p_j^2)} \right) + \mathcal{O}(\epsilon^0)\,.$$

see our Sudakov form factor calculation in the first lecture!

# On-shell matching

To determine the hard function, calculate on-shell amplitudes $|\mathcal{M}_n(\epsilon,\{\underline{p}\})\rangle$ in QCD and effective theory.




In effective theory all loop corrections vanish on-shell, because integrals are scaleless. The IR divergences of the on-shell amplitudes are equal to the UV divergences of $\boldsymbol{C}_n$.

symbolically:

$$\frac{1}{\epsilon_{\rm IR}} = \frac{1}{\epsilon_{\rm UV}} + \underbrace{\left(\frac{1}{\epsilon_{\rm IR}} - \frac{1}{\epsilon_{\rm UV}}\right)}_{\text{soft and collinear loop integrals}}$$

**on-shell amp.** $\boldsymbol{C}_n$

# On-shell matching

The tree-level matrix element of SCET fields are

$$\langle 0|(\chi_j)^a_\alpha(t_j\bar{n}_j)|p_i;a_i,s_i\rangle = \delta_{ij}\,\delta_{a_i a}\,e^{-it_i\bar{n}_i\cdot p_i}\,u_\alpha(p_i,s_i)\,,$$

$$\langle 0|(\mathcal{A}_{j\perp})^a_\mu(t_j\bar{n}_j)|p_i;a_i,s_i\rangle = \delta_{ij}\,\delta_{a_i a}\,e^{-it_i\bar{n}_i\cdot p_i}\,\epsilon_\mu(p_i,s_i)\,.$$

and the convolutions in

$$\mathcal{H}_n^{\text{eff}} = \int dt_1\,\ldots\,dt_n\,\langle O_n(\{\underline{t}\},\mu)|\tilde{\mathcal{C}}_n(\{\underline{t}\},\mu)\rangle\,,$$

produce the Fourier transformed Wilson coefficient $\mathcal{C}_n(\{\underline{p}\},\mu)$ which depend on the large momentum components $\bar{n}_i\cdot p_i$ , or equivalently $s_{ij}$ since

$$s_{ij} = 2\sigma_{ij}\,p_i\cdot p_j = \frac{1}{2}\,\sigma_{ij}\,n_i\cdot n_j\,\bar{n}_i\cdot p_i\,\bar{n}_j\cdot p_j + \mathcal{O}(\lambda)$$

$$p_i^\mu = \bar{n}_i\cdot p_i\frac{n_i^\mu}{2} + \mathcal{O}(\lambda)$$

# Renormalization

So (up to spinors and polarization vectors) the on-shell amplitudes in QCD are equal to bare Wilson coefficients of $n$-jet operators in SCET.

Renormalize

renormalization factor
(minimal subtraction of IR poles)

$$|\mathcal{C}_n(\{\underline{p}\},\mu)\rangle = \lim_{\epsilon\to 0} \boldsymbol{Z}^{-1}(\epsilon,\{\underline{p}\},\mu)\,|\mathcal{M}_n(\epsilon,\{\underline{p}\})\rangle$$

TB, Neubert 2009

- IR poles of scattering amplitudes mapped onto UV poles of n-jet SCET operators
- Multiplicative subtraction, controlled by RG

# Renormalization

The scale dependence of the Wilson is governed by a renormalization group equation

$$\frac{d}{d\ln\mu}\,|\mathcal{C}_n(\{p\},\mu)\rangle = \boldsymbol{\Gamma}(\mu,\{p\})\,|\mathcal{C}_n(\{p\},\mu)\rangle$$

anomalous-dimension matrix

with

$$\boldsymbol{\Gamma}(\{\underline{p}\},\mu) = -\boldsymbol{Z}^{-1}(\epsilon,\{\underline{p}\},\mu)\,\frac{d}{d\ln\mu}\,\boldsymbol{Z}(\epsilon,\{\underline{p}\},\mu)\,.$$

Formal solution

$$\boldsymbol{Z}(\epsilon,\{\underline{p}\},\mu) = \mathbf{P}\exp\left[\int_\mu^\infty \frac{\mathrm{d}\mu'}{\mu'}\,\boldsymbol{\Gamma}(\{\underline{p}\},\mu')\right],$$

# Two-loop result for $\boldsymbol{\Gamma}$

Anomalous dimension is extremely simple:

color charges

anom. dimensions,
known to three-loop order

$$\boldsymbol{\Gamma}(\{\underline{p}\},\mu) = \sum_{(i,j)} \frac{\boldsymbol{T}_i\cdot\boldsymbol{T}_j}{2}\,\gamma_{\rm cusp}(\alpha_s)\,\ln\frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)$$

sum over pairs
i≠j of partons

$(p_i + p_j)^2$

- simple structure, reminiscent of QED
- IR poles determined by color charges and momenta of external partons
- color dipole correlations, like at one-loop order

# $\boldsymbol{Z}$ factor to three loops

Explicit result:

d-dimensional β-function

$$\ln \boldsymbol{Z}(\epsilon, \{\underline{p}\}, \mu) = \int_0^{\alpha_s} \frac{d\alpha}{\alpha} \frac{1}{2\epsilon - \beta(\alpha)/\alpha} \left[ \boldsymbol{\Gamma}(\{\underline{p}\}, \mu, \alpha) + \int_0^{\alpha} \frac{d\alpha'}{\alpha'} \frac{\Gamma'(\alpha')}{2\epsilon - \beta(\alpha')/\alpha'} \right]$$

where

$$\Gamma'(\alpha_s) \equiv \frac{\partial}{\partial \ln \mu} \boldsymbol{\Gamma}(\{\underline{p}\}, \mu, \alpha_s) = -\gamma_{\rm cusp}(\alpha_s) \sum_i C_i$$

Perturbative expansion:

$$\begin{aligned} \ln \boldsymbol{Z} &= \frac{\alpha_s}{4\pi} \left( \frac{\Gamma'_0}{4\epsilon^2} + \frac{\boldsymbol{\Gamma}_0}{2\epsilon} \right) + \left( \frac{\alpha_s}{4\pi} \right)^2 \left[ -\frac{3\beta_0 \Gamma'_0}{16\epsilon^3} + \frac{\Gamma'_1 - 4\beta_0 \boldsymbol{\Gamma}_0}{16\epsilon^2} + \frac{\boldsymbol{\Gamma}_1}{4\epsilon} \right] \\ &+ \left( \frac{\alpha_s}{4\pi} \right)^3 \left[ \frac{11\beta_0^2\, \Gamma'_0}{72\epsilon^4} - \frac{5\beta_0 \Gamma'_1 + 8\beta_1 \Gamma'_0 - 12\beta_0^2\, \boldsymbol{\Gamma}_0}{72\epsilon^3} + \frac{\Gamma'_2 - 6\beta_0 \boldsymbol{\Gamma}_1 - 6\beta_1 \boldsymbol{\Gamma}_0}{36\epsilon^2} + \frac{\boldsymbol{\Gamma}_2}{6\epsilon} \right] + \dots \end{aligned}$$

⇒ exponentiation yields $Z$ factor at three loops!

# Constraints on $\boldsymbol{\Gamma}$

There are many constraints on the anomalous dimension

- Soft-Collinear factorization relates

$$\boldsymbol{\Gamma}_H = \boldsymbol{\Gamma}_S + \sum_i \Gamma_{J_i} \mathbf{1}$$

- Non-abelian exponentiation constrains possible color structures in $\boldsymbol{\Gamma}_S$

- Collinear factorization of amplitudes gives constraints on coefficient functions

Two-loop result is so simple because of these constraints, only at three-loop order new structures appear!

# Factorization constraint on $\mathbf{\Gamma}$

- Operator matrix elements must evolve in the same way as hard matching coefficients, such that physical observables are scale independent

- Factorization of matrix element then implies (with $\Lambda_{ij} = \frac{p_i^2\, p_j^2}{s_{ij}}$):

trivial color structure

$$\mathbf{\Gamma}(s_{ij}) = \mathbf{\Gamma}_s(\Lambda_{ij}) + \sum_i \Gamma_c^i(p_i^2)\mathbf{1}$$

$p_i^2$ dependence must cancel!

  - suggests logarithmic dependence on $s_{ij}$ and $M_i^2$

  - $\mathbf{\Gamma}$ and $\mathbf{\Gamma}_S$ must have same color structure

# Non-abelian exponentiation

The soft function is a matrix element of Wilson lines.

$$\mathcal{S}(\{\underline{n}\},\mu) = \langle 0|\boldsymbol{S}_1(0)\ldots\boldsymbol{S}_n(0)|0\rangle = \exp(\tilde{\mathcal{S}}(\{\underline{n}\},\mu))$$

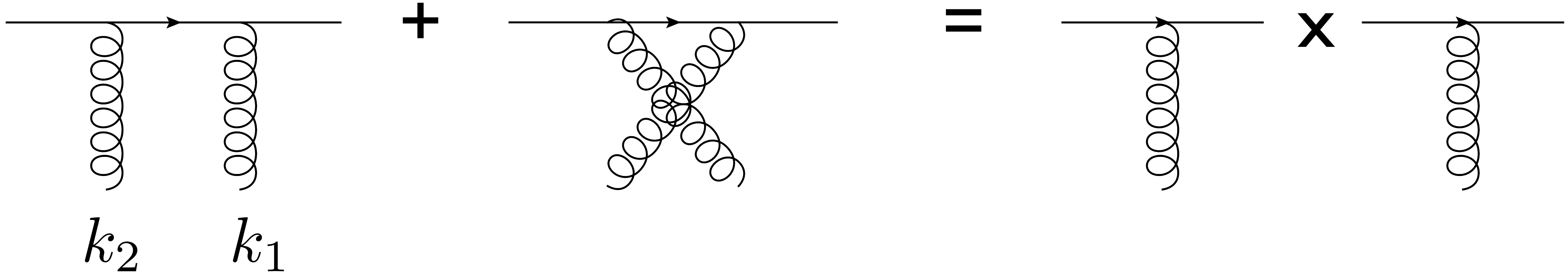
In massive QED, eikonal identities

$$\frac{1}{n\cdot k_1}\frac{1}{n\cdot(k_1+k_2)} + \frac{1}{n\cdot k_2}\frac{1}{n\cdot(k_1+k_2)} = \frac{1}{n\cdot k_1}\frac{1}{n\cdot k_2}$$

imply that such matrix elements exponentiate, i.e. that the exponent $\tilde{\mathcal{S}}$ does not receive higher order corrections!

QCD case is more complicated because the color matrices in the numerator do not commute.

Diagrammatically:



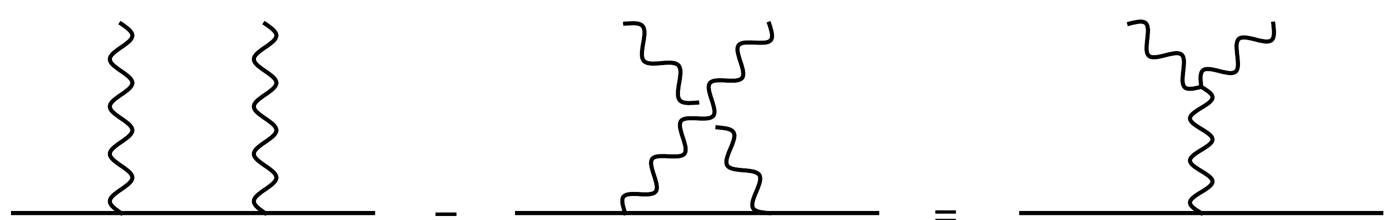

$$\frac{1}{n\cdot k_1}\,\frac{1}{n\cdot(k_1+k_2)}+\frac{1}{n\cdot k_2}\,\frac{1}{n\cdot(k_1+k_2)}=\frac{1}{n\cdot k_1}\,\frac{1}{n\cdot k_2}$$

# Non-abelian exponentiation

Gatheral 1983; Frenkel and Taylor 1984

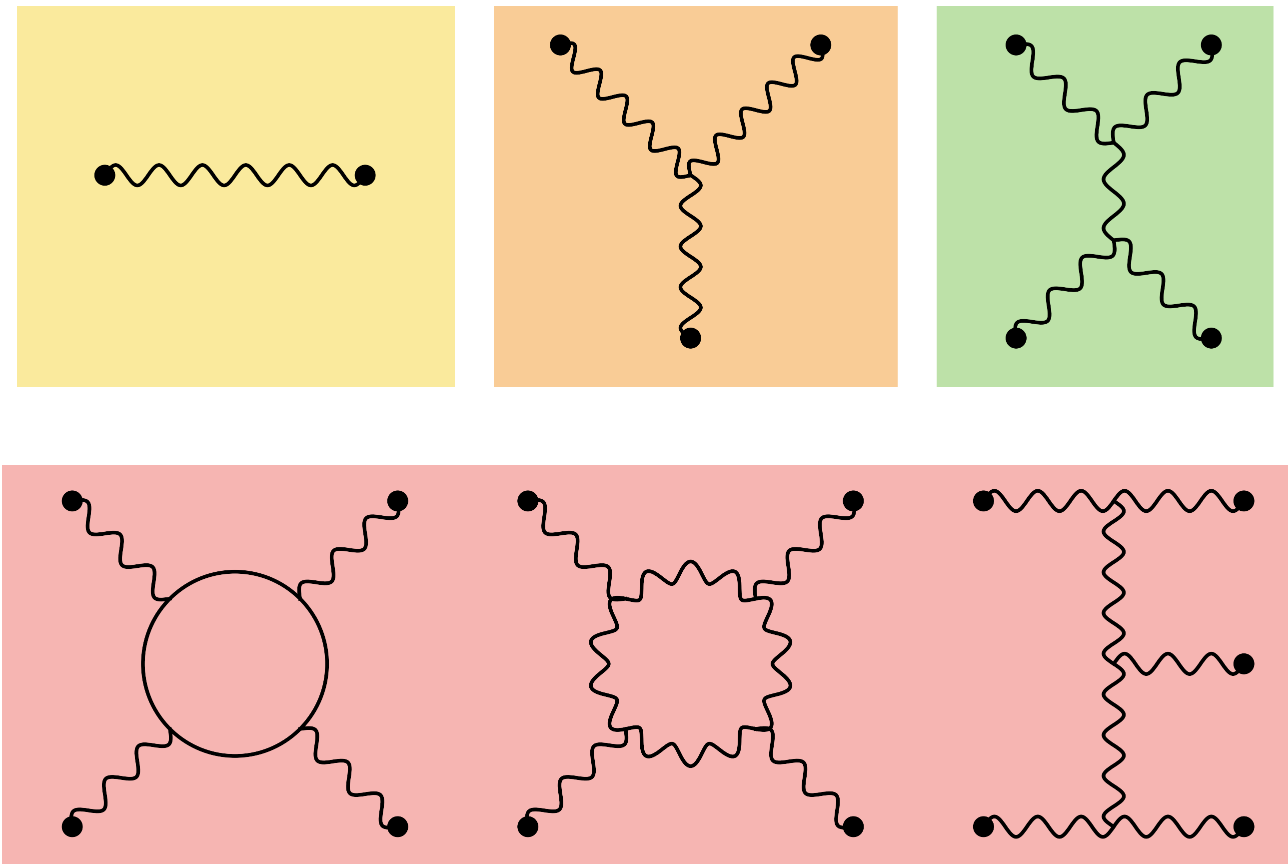
- The exponent $\tilde{S}$ receives contributions only from Feynman diagrams whose color weights are "color-connected" (or "maximally non-abelian")

- Color-weight graphs associated with each Feynman diagram can be simplified using the Lie commutator relation:



$$\boldsymbol{T}^a\boldsymbol{T}^b \quad - \quad \boldsymbol{T}^b\boldsymbol{T}^a \quad = \quad i f^{abc}\boldsymbol{T}^c$$

# Color-connected webs



$$\mathcal{D}_{ij} = \boldsymbol{T}_i^a \boldsymbol{T}_j^a \equiv \boldsymbol{T}_i \cdot \boldsymbol{T}_j \,,$$ $\geq 1$ loops

$$\mathcal{T}_{ijk} = i f^{abc} \left(\boldsymbol{T}_i^a \boldsymbol{T}_j^b \boldsymbol{T}_k^c\right)_+ ,$$ $\geq 2$ loops

$$\mathcal{T}_{ijkl} = f^{ade} f^{bce} \left(\boldsymbol{T}_i^a \boldsymbol{T}_j^b \boldsymbol{T}_k^c \boldsymbol{T}_l^d\right)_+ ,$$ $\geq 3$ loops

$$\mathcal{D}^R_{ijkl} = d_R^{abcd} \, \boldsymbol{T}_i^a \boldsymbol{T}_j^b \boldsymbol{T}_k^c \boldsymbol{T}_l^d \,,$$

$$\mathcal{T}_{ijklm} = i f^{adf} f^{bcg} f^{efg} \left(\boldsymbol{T}_i^a \boldsymbol{T}_j^b \boldsymbol{T}_k^c \boldsymbol{T}_l^d \boldsymbol{T}_m^e\right)_+$$

$\geq 4$ loops

The dots indicate attachments to a leg. Can always symmetrize products of generators.

# Soft-collinear factorization constraint

- With off-shellness $p_i^2 \neq 0$ as IR regulator to define the soft and collinear scales, we obtain:

$$\boxed{\beta_{ij} = L_i + L_j - \ln\frac{\mu^2}{-s_{ij}}}$$

$$\beta_{ij} = \ln\frac{-s_{ij}\,\mu^2}{(-p_i^2)(-p_j^2)}$$

soft log

$$L_i = \ln\frac{\mu^2}{-p_i^2}$$

collinear log

hard log

# Soft anomalous-dimension matrix

Decompositions:

$$\mathbf{\Gamma}(\{\underline{p}\},\mu) = \mathbf{\Gamma}_s(\{\underline{\beta}\},\mu) + \sum_i \Gamma_c^i(L_i,\mu)$$

$$\Gamma_c^i(L_i) = -\Gamma_{\text{cusp}}^i(\alpha_s)\, L_i + \gamma_c^i(\alpha_s)$$

Key equation: Gardi, Magnea, arXiv:0901.1091

$$\boxed{\frac{\partial \mathbf{\Gamma}_s(\{\underline{s}\},\{\underline{L}\},\mu)}{\partial L_i} = \Gamma_{\text{cusp}}^i(\alpha_s)}$$

Suggests linearity in $\beta_{ij}$ and significantly restricts color structures.

# Conformal cross ratios

- Only exception is dependence on conformal cross ratios, which are independent of collinear scales:

$$\beta_{ijkl} = \beta_{ij} + \beta_{kl} - \beta_{ik} - \beta_{jl} = \ln \frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})}$$

Gardi, Magnea 2009

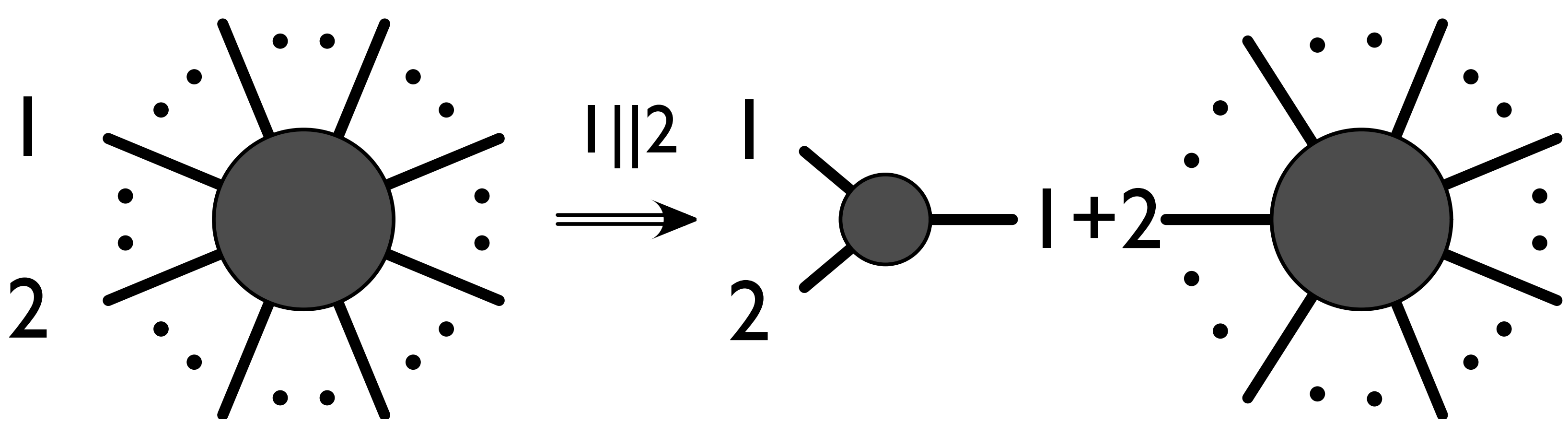
- Dependence on such ratios is restricted by consistency with collinear limits.

# Consistency with collinear limits

- When two partons become collinear, an *n*-point amplitude $\mathcal{M}_n$ reduces to an (*n*-1)-parton amplitude times a splitting function:

Berends, Giele 1989; Mangano, Parke 1991
Kosower 1999; Catani, de Florian, Rodrigo 2003

$$|\mathcal{M}_n(\{p_1,p_2,p_3,\dots,p_n\})\rangle = \mathbf{Sp}(\{p_1,p_2\})\,|\mathcal{M}_{n-1}(\{P,p_3,\dots,p_n\})\rangle + \dots$$




$$\mathbf{\Gamma}_{\rm Sp}(\{p_1,p_2\},\mu) = \mathbf{\Gamma}(\{p_1,\dots,p_n\},\mu) - \mathbf{\Gamma}(\{P,p_3\dots,p_n\},\mu)\big|_{\boldsymbol{T}_P\to\boldsymbol{T}_1+\boldsymbol{T}_2}$$

TB, Neubert 2009

- $\mathbf{\Gamma}_{\rm Sp}$ must be independent of momenta and colors of the partons 3, ..., n

# 4-loop form of $\mathbf{\Gamma}$

TB, Neubert 2019

$$\begin{aligned}
\mathbf{\Gamma}(\{\underline{s}\},\mu) = & \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2}\, \gamma_{\rm cusp}(\alpha_s) \ln \frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)\, \mathbf{1} \\
& + f(\alpha_s) \sum_{(i,j,k)} \mathcal{T}_{iijk} + \sum_{(i,j,k,l)} \mathcal{T}_{ijkl}\, F(\beta_{ijlk}, \beta_{iklj}; \alpha_s) \\
& + \sum_R g^R(\alpha_s) \left[ \sum_{(i,j)} \left( \mathcal{D}^R_{iijj} + 2\mathcal{D}^R_{iiij} \right) \ln \frac{\mu^2}{-s_{ij}} + \sum_{(i,j,k)} \mathcal{D}^R_{ijkk} \ln \frac{\mu^2}{-s_{ij}} \right] \\
& + \sum_R \sum_{(i,j,k,l)} \mathcal{D}^R_{ijkl}\, G^R(\beta_{ijlk}, \beta_{iklj}; \alpha_s) + \sum_{(i,j,k,l)} \mathcal{T}_{ijkli}\, H_1(\beta_{ijlk}, \beta_{iklj}; \alpha_s) \\
& + \sum_{(i,j,k,l,m)} \mathcal{T}_{ijklm}\, H_2(\beta_{ijkl}, \beta_{ijmk}, \beta_{ikmj}, \beta_{jiml}, \beta_{jlmi}; \alpha_s) + \mathcal{O}(\alpha_s^5)\,.
\end{aligned}$$

Three-loop result is fully known Almelid, Duhr, Gardi '16

at four loops only $\ln \mu$ terms and $\gamma^i$ .