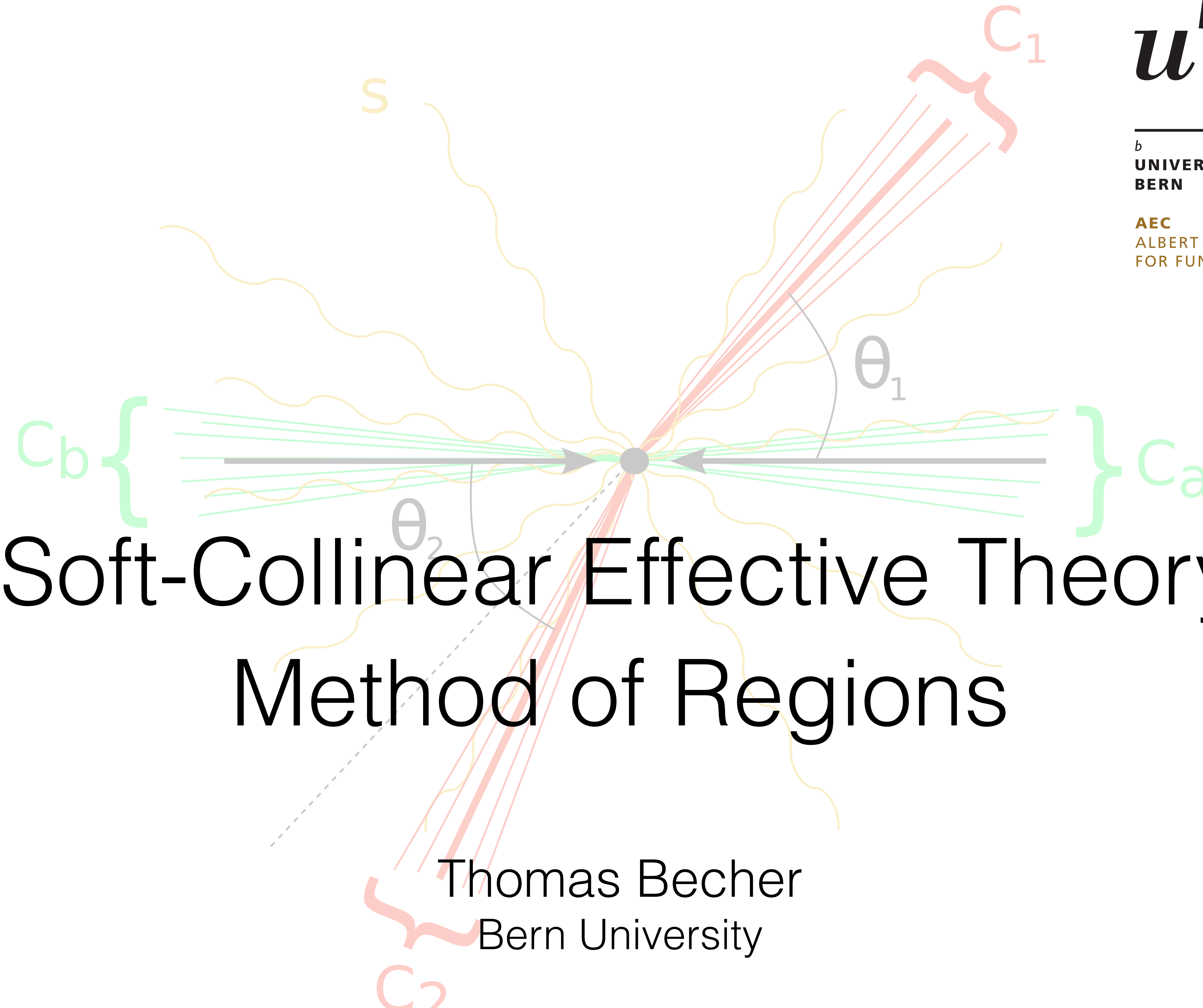

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Soft-Collinear Effective Theory
Method of Regions

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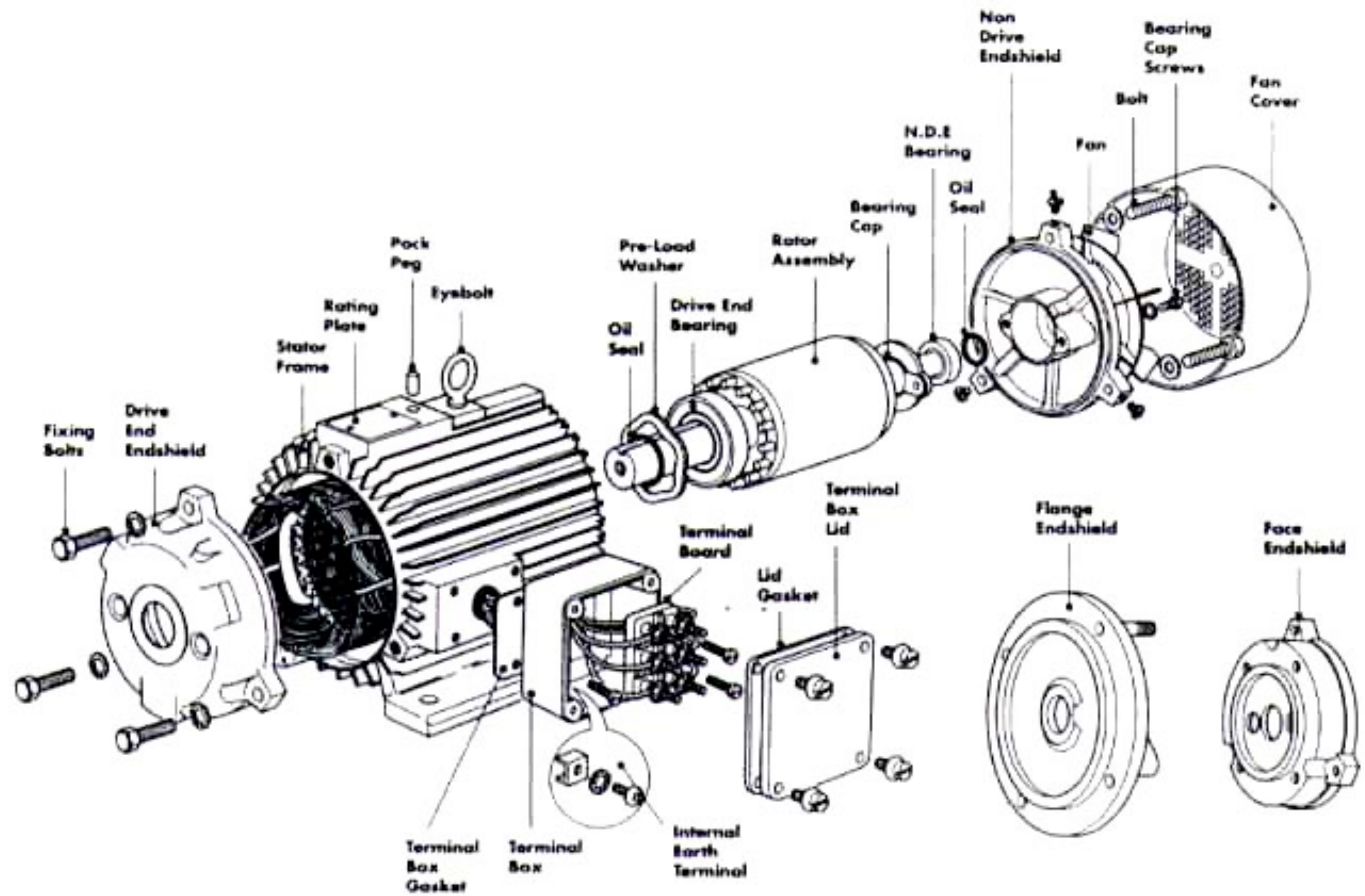


Method of regions

To be ready to set up EFTs with multiple low-energy regions, we discuss the method of regions, a technique to expand loop (and other) integrals around various limits.

- Very useful by itself!
- Low energy regions ↔ EFT fields
- High-energy contributions ↔ Wilson coefficients

The method provides the mathematical foundation for modern EFTs in dimensional regularization.

A simple example

Consider the integral

$$I = \int_0^\infty dk \, \frac{k}{(k^2+m^2)(k^2+M^2)} = \frac{\ln \frac{M}{m}}{M^2-m^2}$$

$$= \frac{\ln \frac{M}{m}}{M^2} \left\{ 1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \ldots \right\} \quad \text{for} \quad m^2 \ll M^2$$

How can we expand the integral *before* performing the integration?

Naive expansion

The naive expansion of the integrand leads to ill-defined expressions:

$$I = \int_0^\infty dk \, \frac{k}{(k^2+m^2)(k^2+M^2)}$$

$$\neq \int_0^\infty dk \, \frac{k}{k^2(k^2+M^2)} \left\{ 1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \dots \right\}$$

IR divergence! For k ~ m, the expansion was not justified.

To be expected, since the result depends non-analytically on m/M...

Regions

Split integration into two regions $m \ll \Lambda \ll M$

$$I = \left[\int_0^\Lambda dk + \int_\Lambda^\infty dk\right] \frac{k}{(k^2+m^2)(k^2+M^2)}$$

(I) + (II)

In the low-energy region (I), $k \sim m \ll M$, expand:

$$I_{(\mathrm{I})} = \int_0^\Lambda dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \int_0^\Lambda dk \frac{k}{(k^2+m^2)M^2}\left\{1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} + \ldots\right\}$$

Here, Λ acts as a UV cut-off.

Regions

In the high-energy region (II), $m \ll k \sim M$, expand:

$$I_{(\mathrm{II})} = \int_{\Lambda}^{\infty} dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \int_{\Lambda}^{\infty} dk \frac{k}{k^2(k^2+M^2)} \left\{ 1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \dots \right\}$$

Λ acts as a IR cut-off.

Result

$$\left. \begin{aligned} I_{(\mathrm{I})} &= -\frac{\Lambda^2}{2M^4} - \frac{1}{M^2} \ln \frac{m}{\Lambda}, \\ I_{(\mathrm{II})} &= +\frac{\Lambda^2}{2M^4} - \frac{1}{M^2} \ln \frac{\Lambda}{M}, \end{aligned} \right\} \quad I = I_{(\mathrm{I})} + I_{(\mathrm{II})} = -\frac{1}{M^2} \ln \frac{m}{M} \quad \checkmark$$

The dependence on the regulator (or better separator) Λ has cancelled among the different regions, and we obtain the correct expansion of the integral.

While the method works, putting hard cut-offs between the momentum regions is impractical for complicated loop integrals.

Fortunately, one can get the same result using dimensional regularization!

Dimensional regularization

Consider our integral in “dimensional regularization”

$$I = \int_0^\infty dk\, k^{-\varepsilon} \frac{k}{(k^2+m^2)(k^2+M^2)}$$

and calculate the contributions obtained by expanding the integral in regions (I) and (II), but without putting an explicit cutoff Λ.

Dimensional regulator ε regulates the IR and UV divergences which appear in the expanded integrals.

Result:

$$I_{(\mathrm{I})} = \int_0^\infty dk\, k^{-\varepsilon} \frac{k}{(k^2+m^2)M^2} \left\{ 1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} + \dots \right\}$$

$$= \frac{m^{-\epsilon}}{M^2} \left\{ \frac{1}{\epsilon} + \mathcal{O}(\epsilon) \right\} + \dots = \frac{1}{M^2} \left\{ \frac{1}{\epsilon} - \ln m + \mathcal{O}(\epsilon) \right\} + \dots$$

UV div.

$$I_{(\mathrm{II})} = \int_0^\infty dk\, k^{-\varepsilon} \frac{k}{k^2(k^2+M^2)} \left\{ 1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \dots \right\}$$

$$= \frac{M^{-\epsilon}}{M^2} \left\{ -\frac{1}{\epsilon} + \mathcal{O}(\epsilon) \right\} + \dots = \frac{1}{M^2} \left\{ -\frac{1}{\epsilon} + \ln M \right\} + \dots$$

IR div.

$$I = I_{(\mathrm{I})} + I_{(\mathrm{II})} = -\frac{1}{M^2} \ln \frac{m}{M} + \dots$$

✓

At first sight, it is surprising that the procedure works. It looks like we are double counting, since we integrate both contributions over the full phase space!

However, note that the two parts scale differently. The low energy integrals behave as $m^{-\varepsilon}$, the high-energy integrals as $M^{-\varepsilon}$.

Full integral for arbitrary ε

(I) + (II)

$$I = \Gamma\left(1 - \frac{\varepsilon}{2}\right)\Gamma\left(\frac{\varepsilon}{2}\right)\frac{m^{-\varepsilon} - M^{-\varepsilon}}{M^2 - m^2}$$

Keeping an explicit cutoff would generate additional $\Lambda^{-\varepsilon}$ pieces, which have to cancel among the two parts since the full integral is Λ independent. E.g.

$$\begin{aligned} I_{(\mathrm{I})} &= \int_0^\Lambda dk\, k^{-\varepsilon}\, \frac{k}{(k^2+m^2)M^2} \left\{1 - \frac{k^2}{M^2} + \dots\right\} \\ &= \left[\int_0^\infty dk - \int_\Lambda^\infty dk\right] k^{-\varepsilon}\, \frac{k}{(k^2+m^2)M^2} \left\{1 - \frac{k^2}{M^2} + \dots\right\} \end{aligned}$$

Cut-off part:

$$\int_\Lambda^\infty dk\, k^{-\varepsilon}\, \frac{k}{(k^2+m^2)M^2} \left\{1 - \frac{k^2}{M^2} + \dots\right\} = \int_\Lambda^\infty dk\, k^{-\varepsilon}\, \frac{k}{k^2\, M^2} \left\{1 - \frac{m^2}{k^2} - \frac{k^2}{M^2} + \dots\right\} \sim \Lambda^{-\varepsilon}$$

Since the $\Lambda^{-\varepsilon}$ pieces cancel in the end, we might as well leave them out from the beginning...

Method of regions

For a review: V.A. Smirnov Springer, Tracts Mod. Phys.177:1-262, 2002

In general, the expansion in dim. reg. is obtained as follows:

- Identify all regions of the integration which lead to singularities in the limit under consideration.
- Expand the integrand in each region and integrate over the *full* phase space.
- Summing the contribution from the different regions gives the expansion of the original integral.

“Problems in the strategy of regions”

V.A. Smirnov, Phys. Lett. B465:226-234,1999

- Make sure the expanded integrals are regularized.
    - Sometimes dimensional regularization is not enough. Can introduce additional analytic regulators or perform subtractions.
- Need to identify all regions of the integration which lead to singularities.
    - There are examples where additional regions appear at higher loop order.

Correspondence with EFT

The soft momentum region directly corresponds to diagrams in the low-energy effective theory.

- In this region $k^\mu \sim m$
- Loop diagrams scale as $m^{-n\varepsilon}$: non-analytic dependence on low-energy scale!

The hard momentum region directly corresponds to matching contributions which are absorbed into Wilson coefficients

- In this region $k^\mu \sim M$
- Loop diagrams scale as $M^{-n\varepsilon}$: non-analytic dependence on high-energy scale!

Matching computations

This yields a very efficient way of performing matching computations in dimensional regularization.

Expand integrands around hard limit

- All EFT loop diagrams vanish. EFT result is tree-level multiplied by Wilson coefficient
- Full theory result is hard part of loop integrals

Wilson coefficients ≡ hard parts of full-theory integrals.

Multi-loop matching computations are always done in this way.

Summary: Method of regions

The expansion of the integral I in the ratio m/M

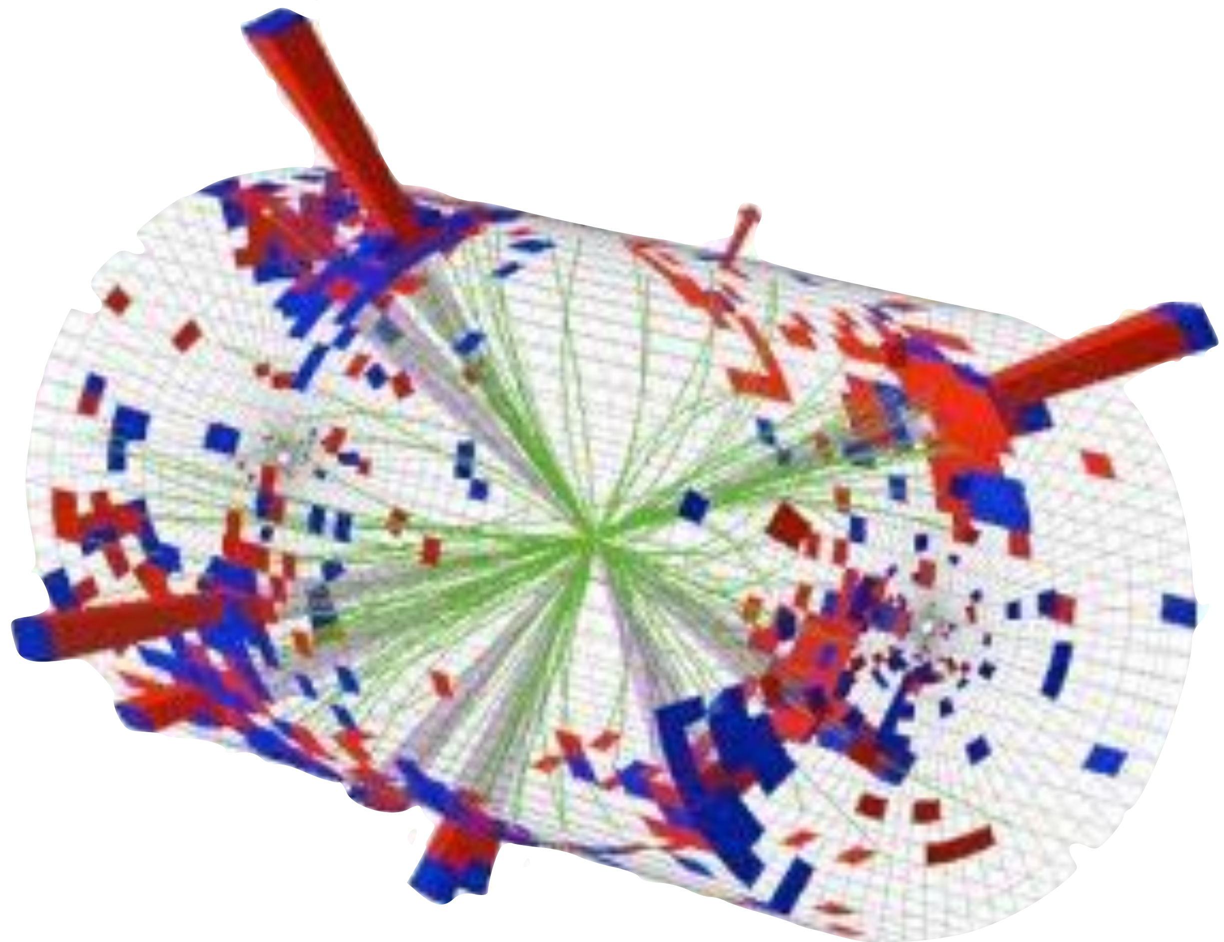
$$I = \int_0^\infty dk\, k^{-\varepsilon}\, \frac{k}{(k^2+m^2)(k^2+M^2)}$$

is obtained by expanding the *integrand* in two regions

I.) soft: $k \sim m$

II.) hard: $k \sim M$

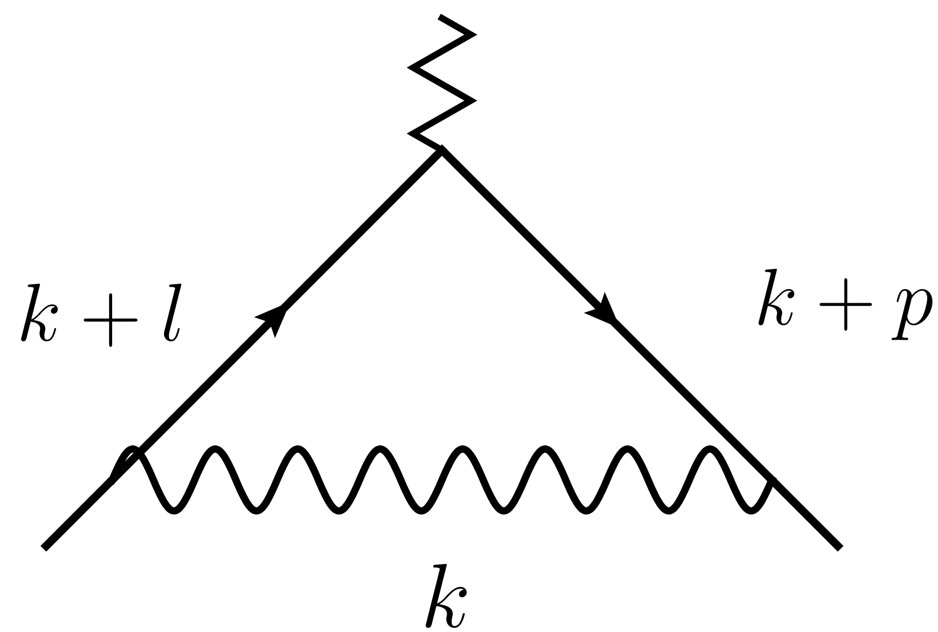
and then integrating both contributions over the *full* phase space.



Momentum Regions in the Sudakov Problem

Application to the Sudakov problem

Let us now perform the expansion in a situation, where particles have large energies, but small invariant masses. Simplest example is the integral



$$L^2 \equiv -l^2 - i0\,, \qquad P^2 \equiv -p^2 - i0\,, \qquad Q^2 \equiv -(l-p)^2 - i0$$

We consider the limit $L^2 \sim P^2 \ll Q^2$.

$$I = i\pi^{-d/2}\mu^{4-d}\int d^dk\,\frac{1}{(k^2+i0)\,[(k+l)^2+i0]\,[(k+p)^2+i0]}$$

We consider the scalar integral *I*, but the same momentum regions appear in tensor integrals.

To obtain the expansion introduce light-like reference vectors in the directions of p and l

$$n_\mu = (1,0,0,1) \qquad \bar{n}_\mu = (1,0,0,-1)$$

$$n^2 = \bar{n}^2 = 0 \qquad n\cdot\bar{n} = 2$$

Any vector can be decomposed as

$$p^\mu = (n\cdot p)\,\frac{\bar{n}^\mu}{2} + (\bar{n}\cdot p)\,\frac{n^\mu}{2} + p_\perp^\mu \equiv p_+^\mu + p_-^\mu + p_\perp^\mu\,,$$

Introduce expansion parameter $\lambda^2 \sim P^2/Q^2 \sim L^2/Q^2$

The different components of p^μ scale differently. Since

$$p^2 = n \cdot p \; \bar{n} \cdot p + p_\perp^2 \sim \lambda^2 Q^2$$

and $p^\mu \approx \frac{1}{2} Q \, n^\mu$, we must have

$$(n \cdot p \, , \, \bar{n} \cdot p \, , \, p_\perp)$$

$$p^\mu \sim \left(\lambda^2 \, , \; 1 \, , \; \lambda\right) Q$$

$$l^\mu \sim \left(1 \, , \; \lambda^2 \, , \; \lambda\right) Q$$

Regions in the Sudakov problem

The following momentum regions contribute to the expansion of the integral

$$(n \cdot k \,,\, \bar{n} \cdot k \,,\, k_\perp)$$

- Hard $(\mathbf{h})$ $\qquad k^\mu \sim (1 \;,\; 1 \;,\; 1\,)\, Q$
- Collinear to p $(\mathbf{c}1)$ $\qquad k^\mu \sim (\lambda^2 \,,\; 1 \;,\; \lambda\,)\, Q$
- Collinear to l $(\mathbf{c}2)$ $\qquad k^\mu \sim (1 \;,\; \lambda^2 \,,\; \lambda\,)\, Q$
- Soft $(\mathbf{s})$ $\qquad k^\mu \sim (\lambda^2 \,,\; \lambda^2 \,,\; \lambda^2)\, Q$

All other possible scalings $(\lambda^a, \lambda^b, \lambda^c)$ lead to scaleless integrals upon expanding. →Exercise

Soft region

Note that the soft region has

$$p_s^2 \sim \lambda^4 Q^2 \sim \frac{L^2 P^2}{Q^2}$$

Interestingly, loop diagrams involve a lower scale than what is present on the external lines. Sometimes scaling $p_s^2 \sim \lambda^4 Q^2$ is called *ultra*-soft.

Implies e.g. that jet-production processes can involve non-perturbative physics, even when the masses of the jets are perturbative.

Low energy regions

In contrast to expansion problems in Euclidean space, we encounter several low-energy regions. Each one is represented by a field in SCET.

Q	QCD	hard	H
M	SCET	collinear	J
M^2/Q	SET	soft	S

Have expanded away small momentum components

$$I_{\rm h} = i\pi^{-d/2}\mu^{4-d}\int d^dk\,\frac{1}{(k^2+i0)\,(k^2+2k_-\cdot l_+ + i0)\,(k^2+2k_+\cdot p_- + i0)}$$
$$= \frac{\Gamma(1+\epsilon)}{2l_+\cdot p_-}\,\frac{\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{\mu^2}{2l_+\cdot p_-}\right)^{\epsilon}$$
$$= \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\ln\frac{\mu^2}{Q^2}+\frac{1}{2}\ln^2\frac{\mu^2}{Q^2}-\frac{\pi^2}{6}\right)+O(\epsilon)\,,$$

IR divergences!

The hard region is given by the on-shell form factor integral.

$$p^\mu \to (\bar{n}\cdot p)\,\frac{n^\mu}{2} \equiv p_-^\mu\,, \qquad l^\mu \to (n\cdot l)\,\frac{\bar{n}^\mu}{2} \equiv l_+^\mu$$

Collinear contribution

$$
\begin{aligned}
I_{\mathrm{c}1} &= i\pi^{-d/2}\mu^{4-d}\int d^dk\,\frac{1}{(k^2+i0)\,(2k_-\cdot l_+ + i0)\,[(k+p)^2+i0]} \\
&= -\frac{\Gamma(1+\epsilon)}{2l_+\cdot p_-}\,\frac{\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{\mu^2}{P^2}\right)^{\epsilon} \\
&= \frac{\Gamma(1+\epsilon)}{Q^2}\left(-\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{\mu^2}{P^2}-\frac{1}{2}\ln^2\frac{\mu^2}{P^2}+\frac{\pi^2}{6}\right)+O(\epsilon)\,.
\end{aligned}
$$

The other collinear contribution $I_{\mathrm{c}2}$ is obtained from exchanging $l \leftrightarrow p$.

Have expanded $(k+l)^2 = 2k_-\cdot l_+ + \mathcal{O}(\lambda^2)$

Scales as $(P^2)^{-\varepsilon}$

Soft contribution

$$I_s = i\pi^{-d/2}\mu^{4-d}\int d^dk\,\frac{1}{(k^2+i0)\,(2k_-\cdot l_+ + l^2 + i0)\,(2k_+\cdot p_- + p^2 + i0)}$$

$$= -\frac{\Gamma(1+\epsilon)}{2l_+\cdot p_-}\,\Gamma(\epsilon)\,\Gamma(-\epsilon)\,\left(\frac{2\mu^2 l_+\cdot p_-}{L^2P^2}\right)^{\epsilon}$$

$$= \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\ln\frac{\mu^2Q^2}{L^2P^2}+\frac{1}{2}\ln^2\frac{\mu^2Q^2}{L^2P^2}+\frac{\pi^2}{6}\right)+O(\epsilon)\,.$$

UV divergences!

Scales as $(\Lambda_{\text{soft}}{}^2)^{-\varepsilon} \sim (P^2\,L^2/Q^2)^{-\varepsilon}$.

Expand $$(k+p)^2 = 2k_+\cdot p_- + p^2 + \mathcal{O}(\lambda^3)$$

Grand total

$$I_{\mathrm{h}} = \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2}{Q^2} + \frac{1}{2}\ln^2\frac{\mu^2}{Q^2} - \frac{\pi^2}{6}\right)$$

$$I_{\mathrm{c}1} = \frac{\Gamma(1+\epsilon)}{Q^2}\left(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu^2}{P^2} - \frac{1}{2}\ln^2\frac{\mu^2}{P^2} + \frac{\pi^2}{6}\right)$$

$$I_{\mathrm{c}2} = \frac{\Gamma(1+\epsilon)}{Q^2}\left(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu^2}{L^2} - \frac{1}{2}\ln^2\frac{\mu^2}{L^2} + \frac{\pi^2}{6}\right)$$

$$I_{s} = \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2 Q^2}{L^2 P^2} + \frac{1}{2}\ln^2\frac{\mu^2 Q^2}{L^2 P^2} + \frac{\pi^2}{6}\right)$$

$$I = I_{\mathrm{h}} + I_{\mathrm{c}1} + I_{\mathrm{c}2} + I_{s} = \frac{1}{Q^2}\left(\ln\frac{Q^2}{L^2}\ln\frac{Q^2}{P^2} + \frac{\pi^2}{3} + O(\lambda)\right)$$

Finite (and correct!)

Cancellations

IR divergences of the hard part are in one-to-one correspondence to UV divergences of the low-energy regions

- True in general: IR divergences of on-shell amplitudes are equal to UV divergences of soft+collinear contributions

The cancellation of divergences involves a nontrivial interplay of soft and collinear log's

$$-\frac{1}{\epsilon}\ln\frac{\mu^2}{P^2}-\frac{1}{\epsilon}\ln\frac{\mu^2}{L^2}+\frac{1}{\epsilon}\ln\frac{\mu^2 Q^2}{L^2 P^2}=-\frac{1}{\epsilon}\ln\frac{\mu^2}{Q^2}$$

- Leads to interesting constraints on IR structure of on-shell amplitudes, see last part of the lecture

Summary

Have identified relevant momentum regions and scaling of fields and expanded QCD diagrams around the relevant limits.

- These are the degrees of freedom in SCET
- Introduce fields with the corresponding scaling

The expanded QCD diagrams are still Feynman diagrams

- Now construct effective Lagrangian whose diagrams are the expanded QCD diagrams!