Introduction to Soft-Collinear Effective Theory

Thomas Becher

Pre-workshop school at SCET 2026

KIAS, February 26–28

- Goal: - Basics for newcomers + the lectures of Chris Lee & Matthias Neubert
  - Subtleties + recent progress for the experts
- Outline:
  - Method of regions
    - One-loop Sudakov form factor
    - Additional regulators, zero bins
    - Exceptional regions: Glauber, Landshoff
    - Cascades of regions, ...
  - Construction of the SCET Lagrangian (at leading power!) + current operator
    - Labelformalism vs. Position space
    - Decoupling and $i\varepsilon$'s.
  - N-jet processes:
    - operators
    - Anomalous dimension
    - IR singularities of amplitudes

- Literature
  - Lecture will follow SCET book 1410.1892 (with Broggio & Ferroglia) and the Les Houches lectures 1803.04310.
  - See slides for original SCET papers and further introductory material.

Method of regions

A characteristic feature of modern EFTs is that we cannot simply integrate out particles, but need to split fields into different "modes" corresponding to different momentum regions, e.g.

$$\phi = \phi_h + \phi_c + \phi_s$$

The hard contribution is integrated out, the low energy contributions are represented by fields in the EFT.

$\rightsquigarrow$ several fields for single particle

A second important feature is that in general not all momentum components scale the same.

⇝ Need reference vectors to isolate different components and organize the expansion

An important tool for the identification of the momentum regions and the foundation of modern EFT in dimensional regularization is the Method of Regions

(Beneke, Smirnov '98)

To apply the method one first needs to identify the momentum regions which contribute to a given process by analyzing loop integrals in the appropriate kinematics. The expansion of the integrals is then obtained by

a.) Expanding the integrand in different momentum regions

b.) Integrating over the full momentum space $\int d^d k$

c.) Adding up the different pieces

Method of regions: Simple Example

Consider Integral (~ Euclidean loop integral in $d=2$)

$$I = \int_0^\infty dk\, k \frac{1}{k^2+M^2}\frac{1}{k^2+m^2} = \frac{\ln(M/m)}{M^2-m^2}$$

with $m \ll M$.

Goal: Expand in $\lambda = m/M \ll 1$ before integration.

Naive expansion of integrand

$$I_{naive} = \int_0^\infty dk\, k \frac{1}{k^2+M^2}\frac{1}{k^2}\left\{1 - \frac{m^2}{k^2} + \dots\right\}$$

IR divergence!

Expansion is not justified when $k \sim m \ll M$.

Need to distinguish two regions

$k \sim M$: high-energy region

$k \sim m$: low-energy region

Quite generally, regions are associated with singularities of the integrand, which arise when particles go on shell

- thresholds (HQET, NRQCD) for massive particles
- soft / collinear limits

In the context of the method of regions not only the singularity, but also how it is approached is relevant, e.g. our regions have the form

$$k \sim \lambda^a M \quad \text{with} \quad \begin{cases} a = 0 \\ a = 1 \end{cases}$$

To expand I, one would introduce Λ with $m \ll \Lambda \ll M$ and split the integral into two regions

"UV regulator"

$$I = \int_0^{\Lambda} dk + \int_{\Lambda}^{\infty} dk \, \frac{k}{(k^2+m^2)(k^2+M^2)}$$

"IR regulator"

$$= I_{(I)} + I_{(II)}$$

perform the appropriate expansion in each region, integrate and add up!

This works and the dependence on Λ drops out after adding the contributions from $I_{(I)}$ and $I_{(II)}$. The introduction of Λ complicates the integral. Simpler regulators for both the UV and IR are

$$I_\varepsilon = \int_0^\infty dk\, k^{1-\varepsilon} \frac{1}{(k^2+m^2)(k^2+M^2)}$$

"dim. reg."

$\sim \int \frac{d^d k}{2\pi}$

or

$$I_\alpha = \int_0^\infty dk\, k \frac{1}{(k^2+m^2)^{1+\alpha}(k^2+M^2)}$$

"analytic regularization"

In standard loop and phase-space integrals we always work in

$d = 4 - 2\varepsilon$, so this regulator is present anyways. However, there are examples ($SCET_{II}$!) where dim. reg. is not sufficient for the phase-space integrals. Similarly, dim. reg. is usually not sufficient for Sudakov problems with masses.

General rule is to put enough regulators such that the expanded integrals in all regions are well defined.

Note that regulators play a double role, analogous the scale Λ used earlier

- Need $\varepsilon > 0$ to regularize UV divs. in low-E integrals
- Need $\varepsilon < 0$ to regularize IR in high-E integrals.

We follow the usual dim. reg. prescription:

i.) Find region $\varepsilon_{min} < \varepsilon < \varepsilon_{max}$ integral $F(\varepsilon)$ is well defined

ii) Analytically continue to <u>all ε</u>

$\rightsquigarrow$ Scaleless integrals are zero!

With the regulators in place, we can now simply expand in each of the two regions and integrate $\int_0^\infty dk$. Adding the contributions and taking $\varepsilon \to 0$ (or $\alpha \to 0$), we recover the expansion of the original integral, see slides.

One way to illustrate why the method works is to first compute the naive expansion to lowest power:

$$I_{(\mathrm{II})} = \int_0^\infty dk\, k^{1-\varepsilon} \frac{1}{k^2 (k^2 + M^2)}$$

Now consider the difference

$$I - I_{(\mathrm{II})} = \int_0^\infty dk\, k \, \frac{1}{k^2+M^2} \times \underbrace{\left\{ \frac{1}{k^2+m^2} - \frac{1}{k^2} \right\}}_{\to 0 \text{ for } k \to \infty}$$

The integrand has only support at low k! We can therefore expand with $k \sim m$:

$$I - I_{(II)}$$

$$= \int_0^\infty dk\, k^{1-\varepsilon} \frac{1}{M^2} \times \left\{ \frac{1}{k^2+m^2} - \underbrace{\frac{1}{k^2}}_{\text{scaleless!} \; = 0} \right\}$$

$$= \int_0^\infty dk\, k^{1-\varepsilon} \frac{1}{M^2} \frac{1}{k^2+m^2}$$

$$= I_{(I)} \quad !$$

So we have demonstrated that

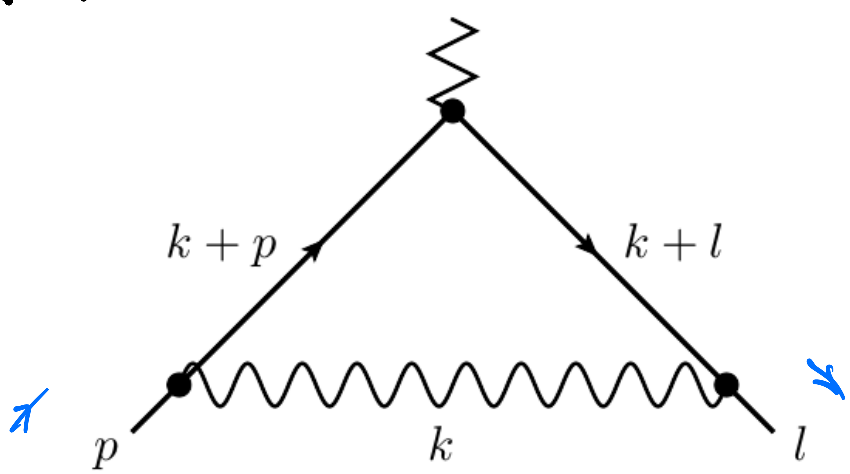
$$I = I_{(I)} + I_{(II)}$$

at leading power in λ.

The deeper reason why this works and why there is no overlap is that the method reduces everything to single-scale integrals. Any further expansions then lead to scaleless integrals.

Sudakov form factor : Method of Region

we now apply this technique to the simplest SCET example, the Sudakov form factor. Based on our results, we then construct an effective theory which implements the expansion.



$$P^2 = -p^2 \; ; \; L^2 = -l^2 \; ; \; Q^2 = -(p-l)^2$$

consider expansion $p^2 \sim l^2 \ll Q^2$

For the moment, we only consider the associated scalar integral, which reads

$$I = \int d^dk \, \frac{1}{k^2 (k+p)^2 (k+l)^2}$$

For the kinematics under consideration p & l have large energies & small virtualit[ies]

To expand around the limit, it is useful to introduce light-like reference vectors

$$n^\mu = (1,0,0,1) \approx p^\mu / p^0$$

$$\bar{n}^\mu = (1,0,0,-1) \approx l^\mu / l^0$$

An arbitrary momentum can be decomposed as

$$q^\mu = n\cdot q \frac{\bar{n}^\mu}{2} + \bar{n}\cdot q \frac{n^\mu}{2} + q_\perp^\mu$$

two orthogonal directions

$n\cdot q_\perp = \bar{n} q_\perp = 0$

$$= q_+^\mu + q_-^\mu + q_\perp^\mu$$

Note that

$$q^2 = q_+\cdot q_- + q_\perp^2$$

Let us introduce an expansion parameter

$$\lambda^2 \sim p^2/Q^2 \sim L^2/Q^2$$

(λ is just a book-keeping device)

Then $p^2 = p_+ p_- + p_\perp^2 \sim \lambda^2 Q^2$

while $(p-\ell)^2 \approx -2\, p_- \cdot \ell_+ \approx Q^2$

The components of p^μ and ℓ^μ therefore scale as:

$$q^\mu \sim (n\cdot q\,,\ \bar{n}\cdot q\,,\ q_\perp)$$

$$p^\mu \sim (\lambda^2\,,\ 1\,,\ \lambda)\,Q$$

$$\ell^\mu \sim (1\,,\ \lambda^2\,,\ \lambda)\,Q$$

Let us now consider different scalings of the loop momentum:

$$(\overset{+}{n\cdot k}\,,\ \overset{-}{\bar{n}\cdot k}\,,\ \overset{\perp}{k_\perp})$$

hard (h): $(1\,,\ 1\,,\ 1)\,Q$

collinear to p^μ (c): $(\lambda^2\,,\ 1\,,\ \lambda)\,Q$

" ℓ^μ ($\bar{c}$): $(1\,,\ \lambda^2\,,\ \lambda)\,Q$

soft (s): $(\lambda^2\,,\ \lambda^2\,,\ \lambda^2)\,Q$

(aka ultrasoft)

Expanding the loop integrand in each region and performing the integrations all of these regions contribute, while all other scalings $k^\mu \sim (\lambda^a, \lambda^b, \lambda^c) Q$ give scaleless integrals upon expanding.

(exercise: pick a scaling and check!)

Let us perform the expansion of the integrand in the different regions.

At leading power

k hard: $(k+p)^2 = (k+p_-)^2 + O(\lambda)$

$(k+\ell)^2 = (k+\ell_+)^2 + O(\lambda)$

k collinear to p: $(k+p)^2 = (k+p)^2$ (no expansion!)

$(k+\ell) = 2k_- \cdot \ell_+ + O(\lambda)$

k soft : $(k+p)^2 = 2p_- \cdot k_+ + p^2 + O(\lambda^3)$

$(k+l)^2 = 2 l_+ k_- + l^2 + O(\lambda^3)$

The expanded loop integrals are

$$I_h = i\pi^{-d/2}\mu^{4-d}\int d^dk \frac{1}{\left(k^2+i0\right)\left(k^2+2k_-\cdot l_+ + i0\right)\left(k^2+2k_+\cdot p_- + i0\right)}$$

$$I_h = \frac{\Gamma(1+\varepsilon)}{2l_+\cdot p_-}\frac{\Gamma^2(-\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{\mu^2}{\underbrace{2l_+\cdot p_-}_{\approx Q^2}}\right)^{\varepsilon} \sim (Q^2)^{-\varepsilon}$$

hard scale

$$I_c = i\pi^{-d/2}\mu^{4-d}\int d^dk \frac{1}{\left(k^2+i0\right)\left(2k_-\cdot l_+ + i0\right)\left[(k+p)^2+i0\right]}$$

$$I_c = -\frac{\Gamma(1+\varepsilon)}{2l_+\cdot p_-}\frac{\Gamma^2(-\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{\mu^2}{P^2}\right)^{\varepsilon} \sim (P^2)^{-\varepsilon}$$

collinear scale

$$\begin{aligned} I_s &= i\pi^{-d/2}\mu^{4-d}\int d^dk \frac{1}{\left(k^2+i0\right)\left(2k_-\cdot l_+ + l^2 + i0\right)\left(2k_+\cdot p_- + p^2 + i0\right)} \\ &= -\frac{\Gamma(1+\varepsilon)}{2l_+\cdot p_-}\Gamma(\varepsilon)\Gamma(-\varepsilon)\left(\frac{2l_+\cdot p_-\mu^2}{L^2P^2}\right)^{\varepsilon}. \end{aligned} \sim (\Lambda_s^2)^{-\varepsilon}$$

Note: soft scale

$$\Lambda_s^2 = \frac{L^2P^2}{Q^2} \ll L^2 \sim P^2$$

After the expansion, these are all single-scale integrals. Note that all of them involve divergences, while the original integral is finite. Let's expand in ε and add up

$$
\begin{aligned}
I_h &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\ln\frac{\mu^2}{Q^2}+\frac{1}{2}\ln^2\frac{\mu^2}{Q^2}-\frac{\pi^2}{6}\right)\\
I_c &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\ln\frac{\mu^2}{P^2}-\frac{1}{2}\ln^2\frac{\mu^2}{P^2}+\frac{\pi^2}{6}\right)\\
I_{\bar{c}} &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\ln\frac{\mu^2}{L^2}-\frac{1}{2}\ln^2\frac{\mu^2}{L^2}+\frac{\pi^2}{6}\right)\\
I_s &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\ln\frac{\mu^2 Q^2}{L^2P^2}+\frac{1}{2}\ln^2\frac{\mu^2 Q^2}{L^2P^2}+\frac{\pi^2}{6}\right)
\end{aligned}
$$

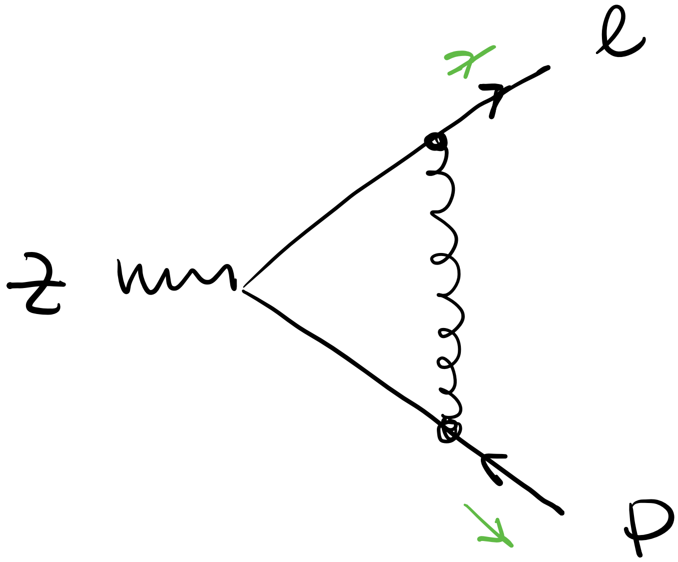
$$
I_{\text{tot}} = \frac{1}{Q^2}\left(\ln\frac{Q^2}{L^2}\ln\frac{Q^2}{P^2}+\frac{\pi^2}{3}\right).
$$

The sum is finite and agrees with the expansion of the original integral!

The cancellation is quite nontrivial since the $1/\varepsilon \ln(\dots)$ divergences involve different scales.

Glauber Phase

Consider the case, where both p^μ & ℓ^μ are outgoing momenta.



This case can be obtained from our result by crossing $p \to -p$ into the final state. Then

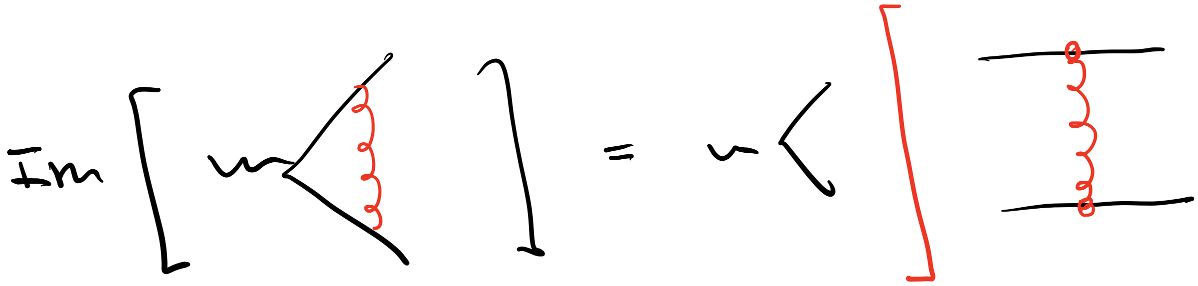
$$Q^2 = -\overbrace{(p+\ell)^2}^{s>0} - i\varepsilon$$

and we pick up imaginary parts from the analytic continuation of the logarithms

$$\ln(Q^2) = \ln(-s - i\varepsilon)$$

$$= \ln(s) - i\pi$$

Note that both the hard region and the soft region have an imaginary part. The imaginary part of the soft function is called the Glauber phase. The phase can be obtained from cutting the diagram



In our off-shell example, the phase is fully contained in the ultrasoft contribution and in e^+e^- it does not contribute to cross sections. (It is lost when performing the decoupling transformation, see later!)

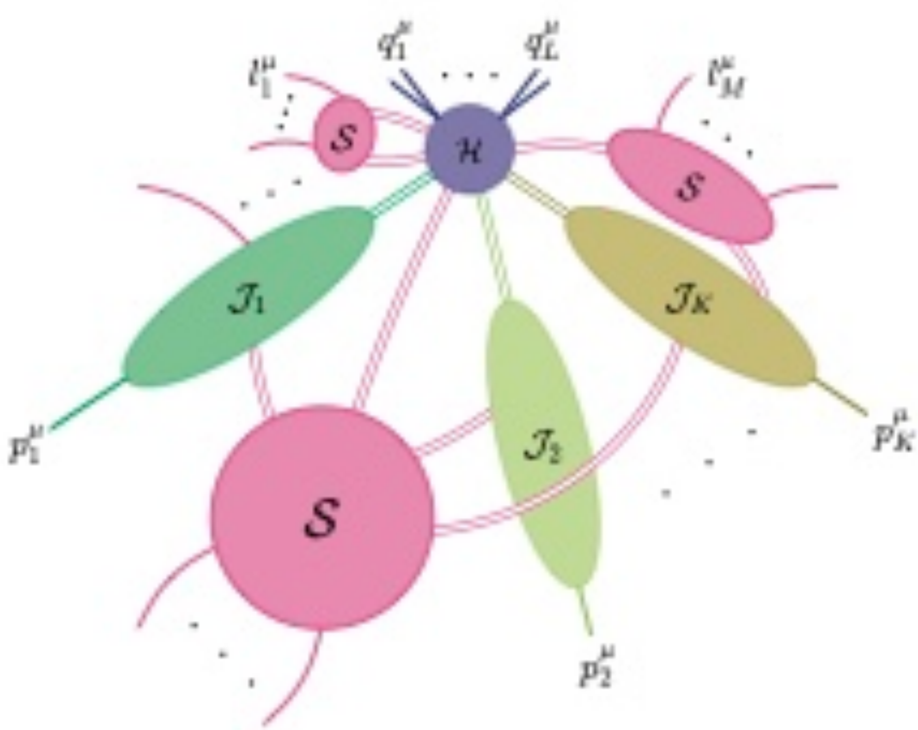
All-order region analysis

The above analysis was carried out at one loop, but we want the EFT to be valid at all orders!

Remarkably, such an all-order analysis is now available [Yao Ma, 2312.14012, 167 pages! ; see also 2211.14845 by Gardi et al., 2601.22144]

This analysis shows that the regions we identified at one loop are the only ones needed to all orders. It also generalizes to the case of large energy in K different directions, relevant in the last part of the lecture.

The structure is the following:



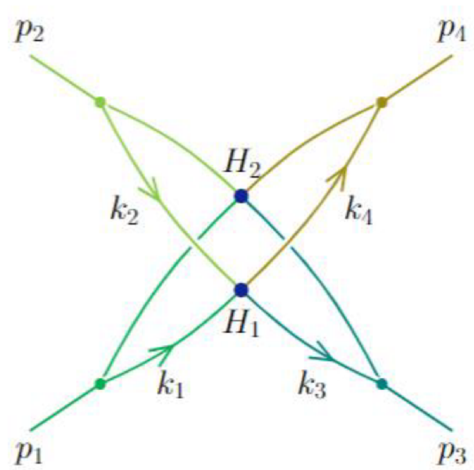
exactly the standard SCET factorization structure!

There is, however, one caveat: The region analysis is performed in Lee-Pomeransky parameter space ($\cong$ Feynman parameter space). The regions that are identified are on the boundary of this parameter space ("facet regions")

However, in some cases also regions inside the parameter space can become relevant ("hidden regions").

For processes with massless partons! where all momenta are outgoing ($e^+e^- \to X$) and cancellations do not arise since all terms in the denominator in parameter space have the same sign.

However, for scattering kinematics they do, even for wide-angle scattering. An example is Landshoff scattering:



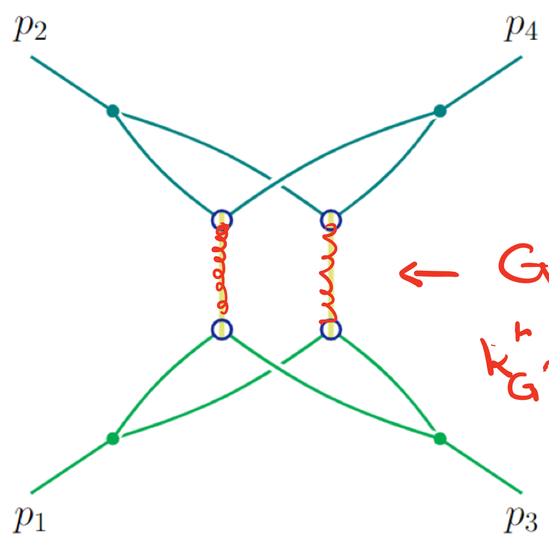
(from talk by Yao Ma)

2211.14845 by Gardi et al.

Collinear modes along all four directions, but two hard scattering centers! Power suppressed in QCD and reproduced by SCET, so not a problem.

The situation is different for forward scattering

$p_2 \parallel p_4$ $\qquad (\lambda^2, 1, \lambda)$

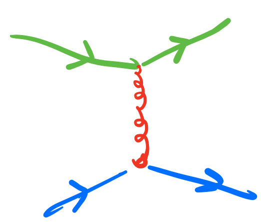


$\leftarrow$ Glauber propagators!

$k_G^\mu \sim (\lambda^2, \lambda^2, \lambda)$

$p_1 \parallel p_3$ $\qquad (1, \lambda^2, \lambda)$

Note that $k_G^2 \approx k_\perp^2 < 0$ is an off-shell propagator. Not a new mode, but SCET for forward scattering needs Glauber operators



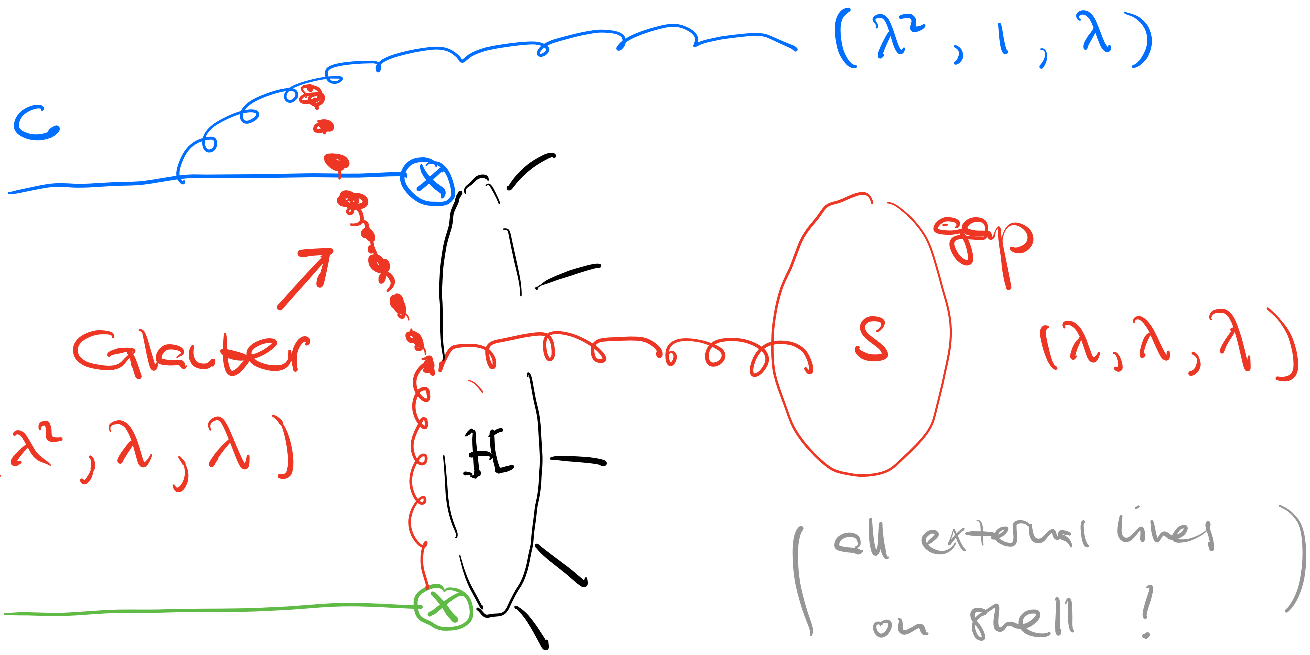
$$\sim \frac{1}{k_\perp^2}$$

This SCET with Glauber operators was constructed by Stewart & Rothstein '16.

An important complication is that all hadron collider cross sections involve some forward scattering due to initial-state collinear emissions.

An explicit recent example are gap-between-jets cross sections, where one vetoes hard jets at central rapidities.

Here one encounters Glauber loops of the form



They are well defined in dim. reg and play an important role in ensuring PDF factorization!

In this introductory lecture, we will not discuss Glauber contributions further, but for hadron collider applications, they need to be taken into account.

These Glauber contributions complicate factorization and resummation. The recent paper 2511.1179 (Banfi, Forshaw, Holguin) claims that such effects give rise to leading-logarithmic contributions not captured by the existing factorization theorem for N-jettiness.

Effective Lagrangian

We now construct an effective theory based on the region expansion of the QCD diagrams: the expanded diagrams are viewed as effective theory diagrams. We introduce fields for the different low-energy regions and construct $\mathcal{L}_{eff}$ whose Feynman rules give the diagrams in the different regions. At tree level, we can substitute

$$
\begin{aligned}
\psi &\to \psi_c + \psi_{\bar{c}} + \psi_s \\
A^\mu &\to A^\mu_c + A^\mu_{\bar{c}} + A^\mu_s
\end{aligned}
\qquad (*)
$$

in the QCD Lagrangian and expand in λ.

To do so, we need to know how the different components of the fields scale.

For the gluon field

$$\langle 0 | T\{ A_\mu^a(x) A_\nu^b(0) \} | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i\delta^{ab}}{k^2} \left\{ -g^{\mu\nu} + \xi \frac{k^\mu k^\nu}{k^2} \right\} \times e^{-ikx}$$

So typically $A_\mu \sim k^\mu$, i.e.

$$(n \cdot A_s, \bar{n} A_s, A_s^\perp) \sim (\lambda^2, \lambda^2, \lambda^2)$$

$$(n \cdot A_c, \bar{n} A_c, A_c^\perp) \sim (\lambda^2, 1, \lambda)$$

For the soft fermion field

$$\langle 0 | T\{ \psi_s(x) \bar{\psi}_s(0) \} | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i\not{k}}{k^2} e^{-ikx}$$

$$\sim \lambda^8 \frac{\lambda^2}{\lambda^4} = \lambda^6$$

$$\rightarrow \quad \psi_s(x) \sim \lambda^3$$

For collinear fermions, the situation is more complicated:

$$\not{k} = k\cdot n \frac{\not{\bar{n}}}{2} + k\cdot\bar{n}\frac{\not{n}}{2} + \not{k}_\perp$$

$$\lambda^2 \qquad\qquad \lambda^0 \qquad\qquad \lambda$$

To separate the different contributions we split the field

$$\psi_c = \xi_c + \eta_c = P_+\psi_c + P_-\psi_c$$

with $P_+ = \frac{\not{n}\not{\bar{n}}}{4}$; $P_- = \frac{\not{\bar{n}}\not{n}}{4}$.

The fulfill $P_+ + P_- = \mathbb{1}$; $P_\pm^2 = P_\pm$.

Then

$$\langle 0| T\{\xi_c(x)\bar{\xi}_c(0)\}|0\rangle = \int\frac{d^4k}{(2\pi)^4} e^{-ikx} \overbrace{\frac{\not{n}\not{\bar{n}}}{4}\frac{i\not{k}}{k^2}\frac{\not{\bar{n}}\not{n}}{4}}^{i\bar{n}\cdot p\frac{\not{n}}{2}\frac{1}{p^2}}$$

$$\sim \lambda^4 \qquad\qquad \cdot\frac{1}{\lambda^2} \sim \lambda^2$$

$\Rightarrow \xi_c \sim \lambda$. Similarly $\eta_c \sim \lambda^2$.

Now that we know how the fields scale, we plug (*) into the QCD action and get

$$S = \underbrace{S_s}_{\text{purely soft}} + S_c + S_{\bar{c}} + \underbrace{S_{s+c}}_{\text{s-c interactions.}} + S_{s+\bar{c}} + \dots$$

The purely soft part

$$S_s = \underbrace{\int d^4x}_{\sim \lambda^{-8}} \; \overbrace{\bar{\psi}_s \, i\underbrace{\not{D}_s}_{i\partial_\mu + gA_s} \psi_s}^{(\lambda^2)^4} - \frac{1}{4} G^a_{s\mu\nu} G_s^{a\mu\nu}$$

has exactly the same form as the usual QCD Lagrangian. All terms are $O(\lambda^0)$

Also the collinear part is a copy of QCD, but we should plug in the decomposition of the fermion field

$$\mathcal{L}_c = (\bar{\xi}_c + \bar{\eta}_c)\left[i n\cdot D_c \frac{\not{\bar{n}}}{2} + i\bar{n}\cdot D \frac{\not{n}}{2} + i\not{D}_\perp \right](\xi_c + \eta_c)$$

$$- \frac{1}{4} G^a_{c\mu\nu} G^{\mu\nu a}_c \,.$$

$$= \bar{\xi}_c\, i n\cdot D \frac{\not{\bar{n}}}{2} \xi_c + \bar{\eta}_c\, i\bar{n}\cdot D \frac{\not{n}}{2} \eta_c + \bar{\eta}_c i\not{D}_{c\perp} \xi_c + \bar{\xi}_c i \not{D}_\perp \eta_c$$

$$- \frac{1}{4} G^a_{c\mu\nu} G^{\mu\nu a}_c \,.$$

This form is inconvenient: $\xi_c \sim \lambda$, $\eta_c \sim \lambda^2$ and the two fields mix. To solve this, one shifts

$$\eta_c \longrightarrow \eta_c - \frac{\not{\bar{n}}}{2} \frac{1}{i\bar{n}\cdot D_c} i\not{D}_{c\perp} \xi$$

large momentum $\sim Q$

to complete the square. This yields

$$\mathcal{L}_c = \bar{\xi}_c \frac{\not{\bar{n}}}{2}\left[i n\cdot D_c + i\not{D}_{c\perp} \frac{1}{i\bar{n}\cdot D_c} i\not{D}_{c\perp} \right] \xi_c$$

$$+\bar{\eta}_c \frac{\not{n}}{2} i\bar{n}\cdot D\, \eta_c - \frac{1}{4} G^a_{c\mu\nu} G_c^{\mu\nu a}$$

Then one integrates out the η_c-field. This leaves a determinant $\det\left(\frac{\not{n}}{2} i\bar{n}\cdot D\right)$. This determinant is trivial. To see this note that it is gauge invariant and manifestly trivial in the gauge $\bar{n}\cdot A = 0$.

Next, let's consider S_{s+c}. Getting the leading-power terms is actually quite simple because:

a.) ψ_s is power suppressed, compared to collinear quarks: no ψ_s at leading power.

b.) $\bar{n}\cdot A_s \ll \bar{n}\cdot A_c$, $A_s^\perp \ll A_c^\perp$. Only $n\cdot A_s \sim n\cdot A_c$ arises at leading power.

Taken together, these imply that the s-c interactions can be obtained by substituting

$$A_c^\mu \to A_c^\mu + n\cdot A_s \frac{\bar{n}^\mu}{2}$$

in $\mathcal{L}_c$. The interaction Lagrangian thus takes the form

$$S_{c+s} = \int d^4x\, \bar{\xi}_c(x) \frac{\not{\bar{n}}}{2} n\cdot A_s(x)\, \xi_c(x)$$

$$+ \text{"gluon terms"}$$

Since this term contains collinear fields and $p_c^\mu + p_s^\mu \sim p_c^\mu \sim (\lambda^2, 1, \lambda)$

$$\rightsquigarrow \qquad x^\mu \sim (1, 1/\lambda^2, 1/\lambda)$$

we can thus perform a derivative expansion, which is the analogue of expanding in the soft momentum, e.g.

$$x_\perp \cdot \partial_\perp \phi_s \sim x^\perp \cdot p_s^\perp \phi_s$$

$$\frac{1}{\lambda} \quad \lambda^2 \quad \sim \lambda$$

$$x_+ \cdot \partial_- \phi_s \sim x_+ \cdot p_{s-} \phi_s$$

$$1 \cdot \lambda^2 \sim \lambda$$

Hence:

$$S_{c+s} = \int d^4x \, \bar{\xi}_c(x) \frac{\not{\bar{n}}}{2} \overbrace{\Big(1 + \underset{\lambda}{x_\perp \cdot \partial_\perp} + \underset{\lambda^2}{x_+ \partial_-} + \underset{\cdots}{\cdots}\Big)}^{\text{Taylor series}}$$

$$\cdot n \cdot A_s(x)\Big|_{x = x_-} \xi_c(x)$$

$$= \int d^4x \, \bar{\xi}_c(x) \frac{\not{\bar{n}}}{2} n \cdot A_s(x_-) \xi_c(x)$$

⇝ In s-c interactions, we must replace $x^\mu \to x_-^\mu$ at leading power.

"Multipole expansion"

Summary:

also contains c+s through n.D.

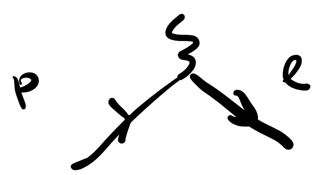
$$\mathcal{L}_{\text{SCET}} = \bar{\psi}_s i\not{D}_s \psi_s + \bar{\xi}_c \frac{\not{\bar{n}}}{2}\left[i n\cdot D + i\not{D}_{c\perp}\frac{1}{i\bar{n}\cdot D_c} i\not{D}_{c\perp}\right]\xi_c - \frac{1}{4}\left(F^{s,a}_{\mu\nu}\right)^2 - \frac{1}{4}\left(F^{c,a}_{\mu\nu}\right)^2$$

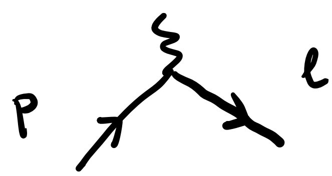
$$+ \mathcal{L}_{\bar{c}} + \mathcal{L}_{\bar{c}+s}$$ ← same form as $\mathcal{L}_c + \mathcal{L}_{c+s}$ but with $n \leftrightarrow \bar{n}$

where $\quad in\cdot D = in\cdot\partial + g\, n\cdot A_c(x) + g\, n\cdot A_s(x_-)$

Vector current in SCET

The final piece of the effective theory is the vector current. At leading order the matching is trivial





$$J^\mu = \bar{\psi}\gamma^\mu\psi \longrightarrow J = \bar{\xi}_{\bar{c}}\gamma^\mu \xi_c$$

$$+ \dots$$

we can simplify this a bit further

$$\bar{\xi}_{\bar{c}}\,\gamma^{\mu}\,\xi_c = \bar{\xi}_{\bar{c}}\Big[\underbrace{n^{\mu}\frac{\bar{\not{n}}}{2}}_{=0} + \underbrace{\bar{n}^{\mu}\frac{\not{n}}{2}}_{=0} + \gamma_{\perp}^{\mu}\Big]\xi_c$$

$$= \bar{\xi}_{\bar{c}}\,\gamma_{\perp}^{\mu}\,\xi_c$$

However, to perform the matching properly, we should write the most general leading-power operator. Since the momentum component $\bar{n}\cdot p$ of a collinear field is large

$$\bar{n}\cdot\partial\,\phi_c \sim \lambda^0 Q\,\phi_c$$

we must allow for operators with arbitrarily many derivatives!

An efficient way of doing so is to use the identities

$$\phi_c(x + t\bar{n}) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (\bar{n}\cdot\partial)^n \phi_c(x)$$

and

$$\int dt\, C(t)\, \phi_c(x + t\bar{n}) = \sum_{n=0}^{\infty} \frac{C_n}{n!} (\bar{n}\cdot\partial)^n \phi_c(x)$$

where $C_n = \int dt\, C(t)\, t^n$.

Instead of introducing infinitely many Wilson coefficients C_n, we smear the field over the light-cone with a function $C(t)$.

In a gauge theory we must make sure that we maintain gauge invariance when smearing the operator, e.g. when writing

$$\bar{\xi}_c(x+t\bar{n})\, U(x+t\bar{n},\, t)\, \xi_c(x)$$

(which is the matrix element defining parton distribution functions!) we need to have a link field U which connects the two fields at different points. We can use the Wilson line

path ordering

matrix field $A^a_\mu \cdot t^a$

$$U_c(x+t\bar{n}, x) = P \exp\left[ig \int_0^t dt'\, \bar{n}\cdot A_c(x+t\bar{n})\right]$$

to achieve this. Under a collinear gauge transformation $V_c(x) = \exp(i\alpha_c^a(x) t^a)$

$$U_c(x+t\bar{n}, x) \to V_c(x+t\bar{n})\, U_c(\dots)\, V_c^\dagger(x).$$

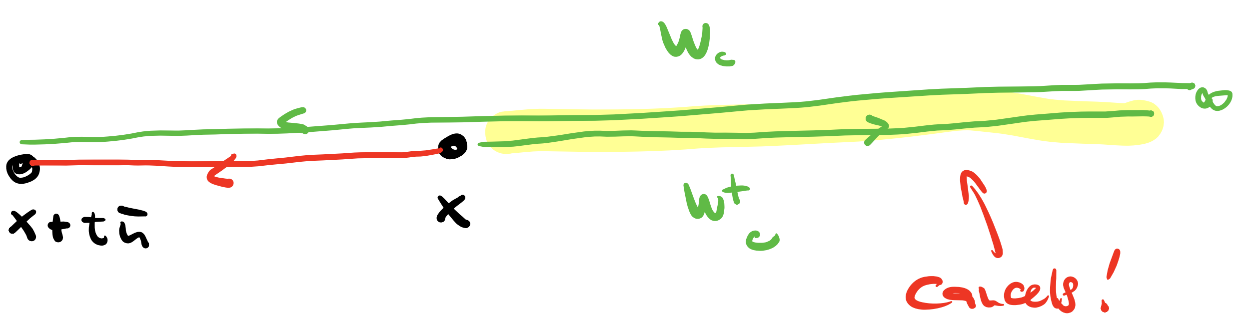
In SCET, one rewrites the link $U_c(x+t\bar{n}, x)$ in terms of the object

$$W_c(x) = U(x, x-\infty\bar{n})$$

in terms of which we can write

$$U(x+t\bar{n}, x) = W_c(x+t\bar{n})\, W_c^\dagger(x)$$

Pictorially:



Using W_c, we can define the building blocks

$$\chi_c(x) = W_c^\dagger(x)\,\xi_c(x)$$

$$\mathcal{A}_c^\mu(x) = W_c^\dagger\left(D_c^\mu W_c\right)$$

which are invariant under collinear gauge transformations which vanish at infinity.

After this long preparation, we are finally ready to write down the leading-power SCET operator:

Wilson coefficient

$$J^\mu(0) = \int ds \int dt\; C_V(s,t)\,\bar{\chi}_{\bar{c}}(sn)\,\gamma_\perp^\mu\,\chi_c(t\bar{n})$$

At tree-level $C_V(s,t) = \delta(s)\delta(t)$. To understand the meaning of C_V, let us take q matrix element

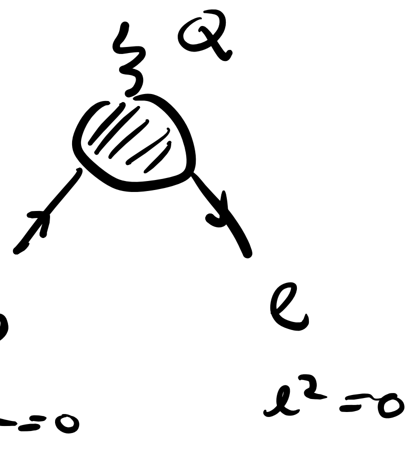
$$\langle q(\ell)|\, J^\mu(0)\,|q(p)\rangle = \int ds \int dt\; C_V(s,t)$$

$$\cdot\; \bar u(\ell)\,\gamma_\perp^\mu\, u(p)\; e^{-is n\ell}\; e^{it\bar n\cdot p}$$

$$= \tilde C_V\big(\overbrace{n\ell\,\bar n p}^{Q^2}\big)\; \bar u(\ell)\,\gamma_\perp^\mu\, u(p)$$

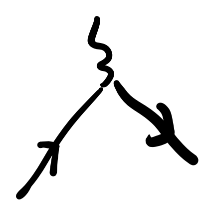
Note that SCET is invariant under the transformation $n \to \alpha n$, $\bar n \to 1/\alpha\, \bar n$. Because of this the coefficient $\tilde C_V$ only depends on $Q^2 = n\ell\,\bar n\cdot p$. To determine $\tilde C_V$ at one computes

QCD



$\equiv$

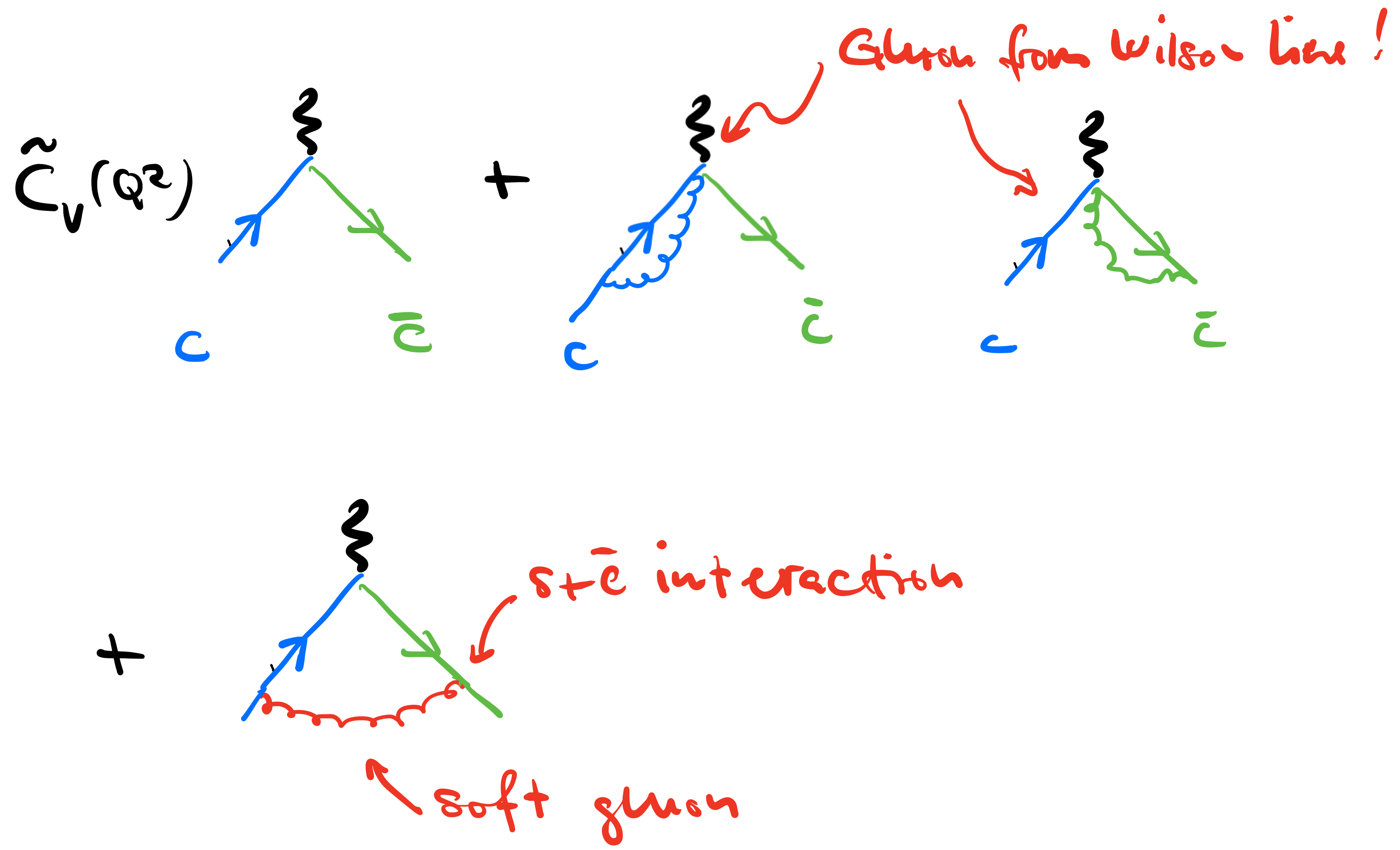
SCET

$\tilde C_V(Q^2)$ 

$+\; 0$ ← collinear & soft loops are scaleless!

The on-shell form factor directly corresponds to $\tilde{C}_V$!

It is now an interesting exercise to verify that all regions occurring in the diagrammatic analysis using the method of regions are fully reproduced by SCET. This is indeed the case, see 1410.1892



Decoupling transformation & factorization

Next, we'll proceed as in the QED case:

we'll perform a field redefinition

multipole expansion!

$$\xi_c(x) = S_n(x_-)\, \xi_c^{(0)}(x)$$

$$A_c^\mu = S_n(x_-)\, A_c^{(0)\mu}\, S_n^\dagger(x_-)$$

with the soft Wilson line

$$S_n(x) := \mathbb{P} \exp\left[ig \int_{-\infty}^{0} ds\, n \cdot A_s(x+sn) \right]$$

matrix $A_\mu^a \cdot t^a$

path ordering for the matrices

The result is the same as for the QED case: the soft field decouples from the leading-power $\mathcal{L}_{c+s}$:

$$\bar{\xi}_c\, in\cdot D\, \frac{\not{\bar{n}}}{2}\, \xi_c = \bar{\xi}_c^{(0)}\, in\cdot D_c\, \xi_c^{(0)}$$

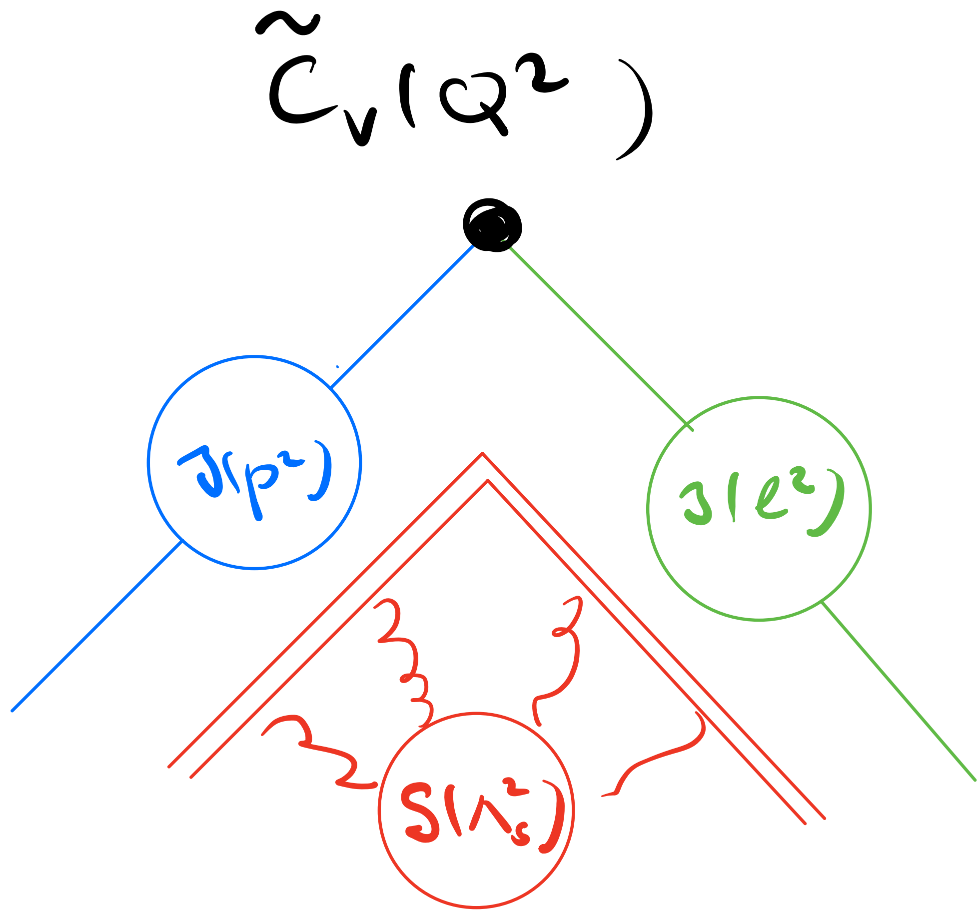
Also, again as in the QED example, the soft Wilson lines appear in the operator:

$$J^\mu(0) = \int ds \int dt\, C_V(s,t)$$

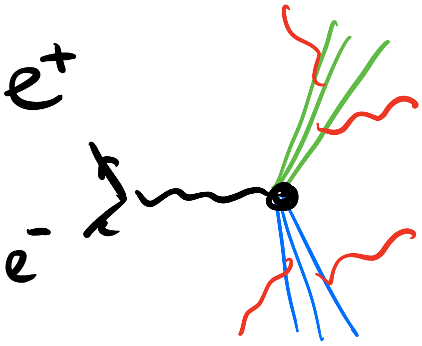
$$\underbrace{\bar{\chi}_{\bar{c}}^{(0)}(sn)}\ \underbrace{S_{\bar{n}}^{\dagger}(0)\,\gamma_\perp^\mu\, S_n(0)}_{\uparrow\ x_- = \bar{n}\cdot x\,\frac{n^\mu}{2}}\ \underbrace{\chi_c^{(0)}(t\bar{n})}$$

After the decoupling, the 3 different fields no longer interact: We have factorized the Sudakov form factor.

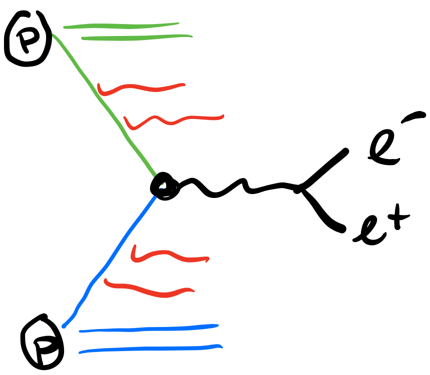
Schematically:



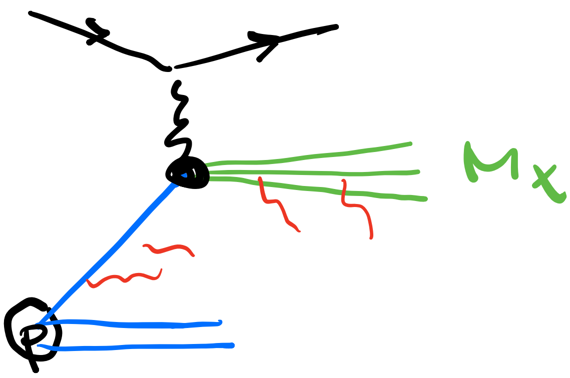
Unfortunately we don't have time to discuss applications of SCET beyond the Sudakov form factor, however, the analysis of the vector form factor forms the basis of many applications, e.g.



Event-shapes in 2-jet production (e.g. thrust)



Drell-Yan process ($pp \to e^+e^- + X$) near threshold.



Deep Inelastic Scattering ($e^-p \to e^- + X$) for small M_X

Let us discuss a subtlety related to the decoupling. We have used the Wilson line

$$S_n(x) := \mathbb{P} \exp\left[ig \int_{-\infty}^{0} ds\, n \cdot A_s(x + sn) \right]$$

to perform the decoupling. Physics-wise this corresponds to soft emissions off an incoming quark. We can also consider outgoing Wilson lines

$$\bar{S}_n(x) = \mathbb{P} \exp\left[ig \int_{0}^{\infty} ds\, n \cdot A_s(x + sn) \right]$$

and decouple with

$$\xi(x) \longrightarrow \bar{S}_n^{\dagger}(x_-)\, \xi^{(0)}(x)$$

The only difference is the $i\varepsilon$ in the soft propagators

$$S_n(x) \longrightarrow -g_s \frac{n^\mu t^a}{n\cdot k + i\varepsilon}$$

$$\underbrace{\bar{S}_n^\dagger(x) \longrightarrow -g_s \frac{n^\mu t^a}{n\cdot k - i\varepsilon}}_{\text{Feynman rule for gluon with inc. mom. } k^\mu}$$

We must make sure to reproduce the correct $i\varepsilon$'s present in QCD and should therefore use the appropriate Wilson lines for the decoupling (outgoing for outgoing particles). Alternatively, one can subtract the phases in the soft function and add them via Glauber - SCET.

Label formalism

In HQET one usually splits $\lambda = \frac{\Lambda_{QCD}}{m_Q}$

$$p^\mu = \underbrace{m_Q v^\mu}_{\text{large } O(1)} + \underbrace{r^\mu}_{\text{small } O(\lambda)}$$

and redefines

$$\psi_Q(x) = e^{-i m_Q v \cdot x} h_v(x) .$$

The field h_v then carries the residual momentum and is slowly varying $x^\mu \sim \frac{1}{\lambda}$.

The label formalism does the same for the SCET fields

$$\chi(x) = \sum_q e^{\overset{\lambda^0}{-iq_- x_+}\ \overset{\lambda^1}{-iq_\perp x_\perp}} \underset{\text{label}}{\chi_q}(x)$$

$x^+ \sim \frac{1}{\lambda^2}$

In contrast to the velocity in HQET, the large momentum component of χ can change, e.g. in collinear splittings.

However, the total label momentum is conserved. To implement this one introduces the label operator

$$\mathcal{P}^\mu \chi_q(x) = q^\mu \chi_q(x)$$

We can rewrite the SCET Lagrangian in terms of the gauge invariant building blocks χ by inserting $WW^\dagger$ between fields.

This leads to

$$\mathcal{L}_c = \bar{\chi}\,\frac{\not{\bar{n}}}{2}\left(in\cdot\partial + n\cdot A\right)\chi$$

$$+ \bar{\chi}\left(i\not{\partial}_\perp + \not{A}_\perp\right)\frac{1}{i\bar{n}\cdot\partial}\left(i\not{\partial}_\perp + \not{A}_\perp\right)\chi$$

Note:

$$\frac{1}{\bar{n}\cdot D_c} = WW^\dagger\,\frac{1}{\bar{n}\cdot D_c}\,WW^\dagger$$

$$= W\big(\underbrace{W^\dagger i\bar{n}\cdot D W_c}_{i\bar{n}\partial}\big)^{-1}W^\dagger = W\,\frac{1}{i\bar{n}\partial}\,W^\dagger$$

Rewriting this in terms of the label fields, one obtains

$$\mathcal{L}_c = \bar{\chi}_q \frac{\bar{n}\!\!\!/}{2}(in\cdot\mathcal{P} + n\cdot A_k)\chi_{q'}$$
$$+ \bar{\chi}_q(i\mathcal{P}\!\!\!/_\perp + A\!\!\!/_{\perp k})\frac{1}{i\bar{n}\cdot\mathcal{P}}(i\mathcal{P}\!\!\!/_\perp + A\!\!\!/_{\perp k'})\chi_{q'}$$

where we suppress the label sums and the phase factors ensuring label momentum conservation.

It is interesting to also rewrite the current operator in the label formalism

$$J^\mu(0) = \int ds \int dt \, C_V(s,t)$$

$$\bar{\chi}_c(s\bar{n}) \, \gamma_\perp^\mu \, \chi(tn)$$

$$= \sum_{q,k} \int ds \int dt \, C_V(s,t)$$

$$\bar{\chi}_{c,q}(0) \underbrace{e^{i\bar{n}q s} \, \gamma_\perp^\mu \, e^{-inkt}}_{e^{i\bar{n}\cdot\mathcal{P}^\dagger s} \, \gamma_\perp^\mu \, e^{-in\cdot\bar{\mathcal{P}} t}} \chi_{\bar{c},k}(0)$$

$$= \sum_{q,k} \tilde{C}_V(\bar{n}\cdot\mathcal{P}^\dagger, n\cdot\bar{\mathcal{P}})$$

$$\times \; \bar{\chi}_{c,q}(0)\, \gamma_\perp^\mu\, \chi_{\bar{c},h}(0)$$

As these examples illustrate, at leading power it is easy to translate between the formalisms.

SCET for N-jet processes

$\rightsquigarrow$ see slides!