Two Loop Anomalous Dimension for Non Global Logarithms

A deep dive into Scheme Choices and Clustering Effects

Based on work with Thomas Becher and Nicolas Schalch

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Part 1: Scheme Dependence

Why $\overline{\mathrm{MS}}$ is Unsuitable

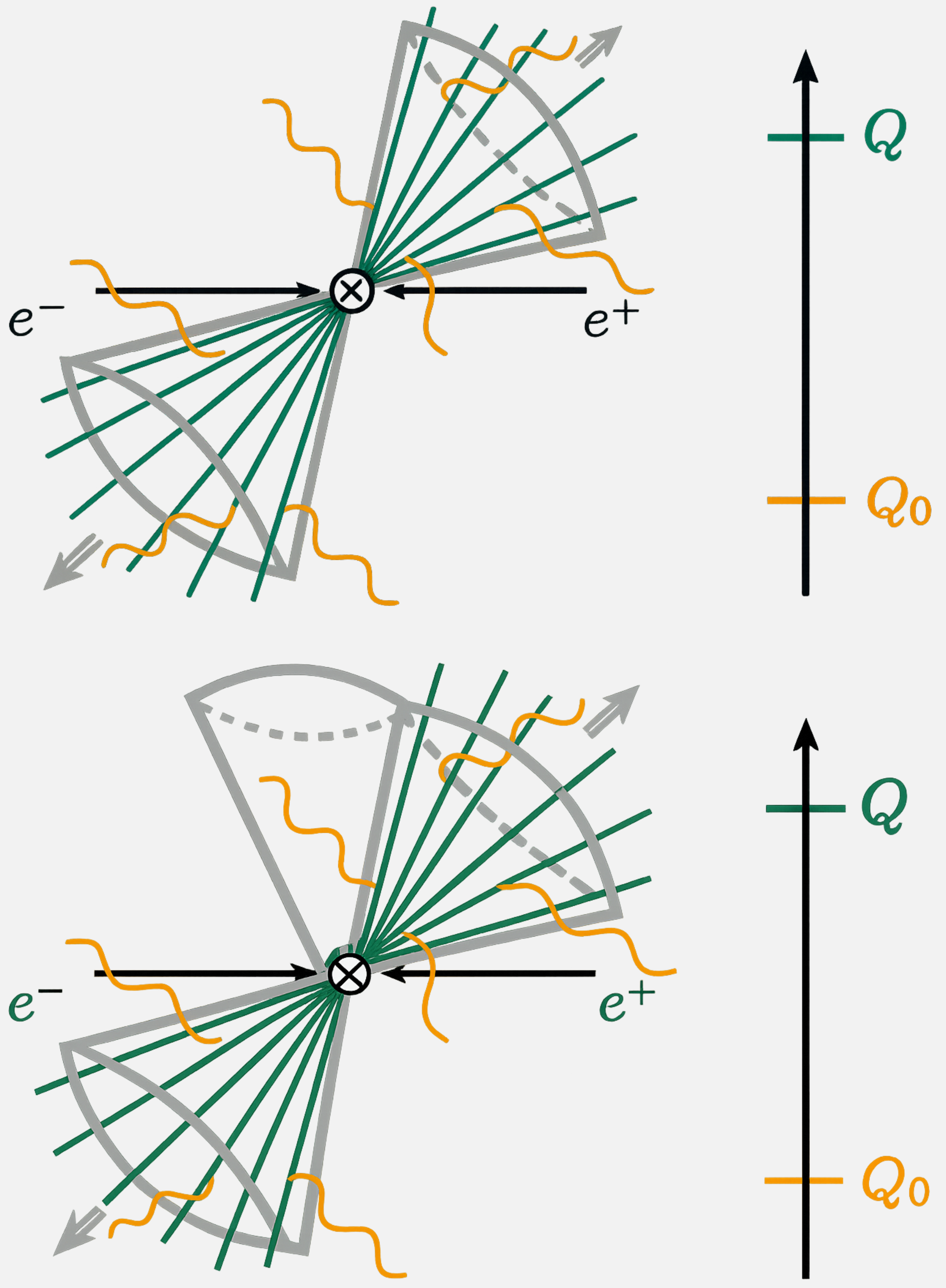
Exclusive jet cross-sections

Gaps between jets

- We are interested in cross sections of the form

$$\sigma\left(Q_0\right)=\frac{1}{2Q^2}\sum_{m=M}^{\infty}\prod_{i=1}^{m}\int\left[dp_i\right]\left|\mathcal{M}_m(\{\underline{p}\})\right|^2\delta\left(Q-E_{\mathrm{tot}}\right)\delta^{(d-1)}\left(\vec{p}_{\mathrm{tot}}\right)\Theta\left(Q_0-2E_{\mathrm{out}}\right)$$

- The energy veto $\Theta\left(Q_0-2E_{\mathrm{out}}\right)$ introduces non-global logarithms $\alpha_S\log\left(\frac{Q}{Q_0}\right)$
- What does "out" mean?
- Fixed cone cross section: "out" depends only on the hard scale dynamics,
  ⇒**Correlated** soft emissions do not exponentiate! NGLs!
- Sequential clustering: "out" also depends on the soft scale dynamics.
  ⇒**All** soft emissions do not exponentiate! Clustering Logs!



Factorization and The Soft Function

- The factorization formula reads $\sigma(Q,Q_0) = \sum_{m=M}^{\infty} \langle \boldsymbol{\mathcal{H}}_m(\{\underline{n}\},\{\underline{z}\},Q,\mu) \otimes_z \boldsymbol{\mathcal{S}}_m(\{\underline{n}\},\{\underline{z}\},Q_0,\mu)\rangle$
- The all-order definition of the soft function reads

$$\boldsymbol{\mathcal{S}}_m(\{\underline{n}\},\{\underline{z}\},Q_0) = \sum\!\!\!\!\!\!\!\int_X \langle 0|\boldsymbol{\mathcal{S}}_1^\dagger(n_1)\dots\boldsymbol{\mathcal{S}}_m^\dagger(n_m)|X\rangle\langle X|\boldsymbol{\mathcal{S}}_1(n_1)\dots\boldsymbol{\mathcal{S}}_m(n_m)|0\rangle\theta(Q_0-2E_{\rm out})$$

Directions of hard partons ($\{\underline{n}\}$)

Energy fractions of hard partons ($\{\underline{z}\}$)

- At one loop for fixed cones, we find

$$\frac{\alpha_s}{4\pi}\boldsymbol{S}_m^{(1)}(\{\underline{n}\},Q_0) = -g_s^2\sum_{(ij)} \boldsymbol{T}_{i,L}\cdot\boldsymbol{T}_{j,R}\,\tilde{\mu}^{4-d}\int\frac{d^dq}{(2\pi)^{d-1}}W_{ij}^q\,\delta(q^2)\theta(q^0)\,\theta(E_{\rm out}-v\cdot q)\Theta_{\rm out}(n_q)$$

$$W_{ij}^q = \frac{n_{ij}}{n_{iq}\,n_{jq}}$$

This vector allows us to keep track of Lorentz covariance. In th labe frame it reads $v=(1,0,0,0)^T$

One-Loop Soft Function: Scheme Dependence

In a general scheme, the renormalized soft function is given by

$$\boldsymbol{\mathcal{S}}^{\mathrm{RS}(1)} = \lim_{\epsilon\to 0}\left[\boldsymbol{\mathcal{S}}^{(1)} - \frac{1}{2\epsilon}\boldsymbol{\Gamma}^{\mathrm{RS}(1)}\hat{\otimes}\mathbf{1}\right]$$

For simplicity, we assume that θ_{out} can be written as $\theta_{\text{out}}(n_{qv}, n_{qi}, n_{qj}, n_q \cdot n_\perp)$. This is certainly true in 4 dimensions.

In the dipole rest framehere, n_q reads

$$n_q \equiv n_q(\theta,\phi,\chi) = (1, \underbrace{\cos\theta}_{i,j}, \underbrace{\sin\theta\cos\phi}_{v}, \underbrace{\sin\theta\cos\phi\cos\chi}_{\perp}, \sin\theta\sin\chi\,\hat{n}_{d-4})$$

and the anomalous dimension acts on the trivial (0-loop) soft function as

$$-\frac{1}{2\epsilon}\boldsymbol{\Gamma}^{\mathrm{RS}(1)}\hat{\otimes}\mathbf{1} = -\frac{2}{\epsilon}\sum_{(ij)}\boldsymbol{T}_i\cdot\boldsymbol{T}_j\,\mathcal{S}_{ij}^{\mathrm{RS}}$$

$$\mathcal{S}_{ij}^{\mathrm{RS}} = \int [d\Omega_q]\,\underbrace{W_{ij}^q\left(\frac{\sin^2\theta}{4n_{vq}^2}\right)^{a\epsilon}(4\sin^2\phi)^{b\epsilon}}_{\text{Modified dipole}}\;\underbrace{\Theta_{\text{out}}^{\mathrm{RS}}(n_q)}_{\text{Modified veto}}$$

$$\int [d\Omega_q] = \frac{e^{\epsilon\gamma_E}\Omega_{d-4}}{(4\pi)^{1-2\epsilon}}\int_0^\pi d\theta\,(\sin\theta)^{1-2\epsilon}\int_{-\pi}^{\pi} d\phi\,(\sin\phi)^{-2\epsilon}\int_0^{\frac{\pi}{2}} d\chi\,(\sin\chi)^{-1-2\epsilon}\,(n_{vq})^{2\epsilon}$$

One-Loop Soft Function: Scheme Dependence

$$-\frac{1}{2\epsilon}\boldsymbol{\Gamma}^{\mathrm{RS}(1)}\hat{\otimes}\mathbf{1} = -\frac{2}{\epsilon}\sum_{(ij)}\boldsymbol{T}_i\cdot\boldsymbol{T}_j\,\mathcal{S}_{ij}^{\mathrm{RS}} \qquad \mathcal{S}_{ij}^{\mathrm{RS}} = \int[d\Omega_q]\,W_{ij}^{q}\left(\frac{\sin^2\theta}{4n_{vq}^2}\right)^{a\epsilon}\left(4\sin^2\phi\right)^{b\epsilon}\,\Theta_{\mathrm{out}}^{\mathrm{RS}}(n_q)$$

The renormalized soft function becomes

$$\boldsymbol{\mathcal{S}}^{\mathrm{RS}(1)} = \sum_{(ij)}\boldsymbol{T}_i\cdot\boldsymbol{T}_j\left[4\ln\left(\frac{\mu}{Q_0}\right)\mathcal{S}_{ij}^{[0]} + \Delta\mathcal{S}_{ij}\right]$$

with

$$\Delta\mathcal{S}_{ij} = \frac{1}{4\pi}\int_0^{\pi}d\theta\int_0^{2\pi}d\phi\frac{4}{\sin^2\theta}\int_0^{\frac{\pi}{2}}d\chi\left[\frac{1}{\sin\chi}\left(\theta_{\mathrm{out}}^{\mathrm{RS}}\left(n_q\left(\theta,\phi,\chi\right)\right)-\theta_{\mathrm{out}}\left(n_q\left(\theta,\phi,\chi\right)\right)\right)\right.$$
$$\left.+\frac{2}{\pi}\theta_{\mathrm{out}}\left(n_q\left(\theta,\phi,0\right)\right)\left[a\ln\left(\frac{4}{\sin^2\theta}\right)-b\ln\left(4\sin^2\phi\right)\right]\right]$$

in the dipole rest frame.

One-Loop Soft Function: Scheme Options

$$\Delta\mathcal{S}_{ij} = \frac{1}{4\pi}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \frac{4}{\sin^2\theta}\int_0^{\frac{\pi}{2}} d\chi \Bigg[\frac{1}{\sin\chi}\left(\theta_{\rm out}^{\rm RS}\left(n_q\left(\theta,\phi,\chi\right)\right) - \theta_{\rm out}\left(n_q\left(\theta,\phi,\chi\right)\right)\right) + \frac{2}{\pi}\theta_{\rm out}\left(n_q\left(\theta,\phi,0\right)\right)\left[a\ln\!\left(\frac{4}{\sin^2\theta}\right) - b\ln\!\left(4\sin^2\phi\right)\right]\Bigg]$$

Scheme	a	b	$\theta_{\rm out}^{\rm RS}(\theta,\phi,\chi)$
$\overline{\rm MS}$	1	1	$\theta_{\rm out}(\theta,\phi,0)$
BS	1	1	$\theta_{\rm out}(\theta,\phi,\chi)$
LS	1	0	$\theta_{\rm out}(\theta,\phi,\chi)$
$\overline{\rm XS}$	0	0	$\theta_{\rm out}(\theta,\phi,\chi)$

$\overline{\rm MS}$-scheme requires the introduction of extra angles in the shower!

The Maximal Subtraction Scheme

In the maximal subtractions $\overline{\text{XS}}$ scheme, the renormalized soft function is only a logarithm (for energy vetos)

$$\boldsymbol{\mathcal{S}}_m^{(1)\overline{\text{XS}}} = 4\ln\left(\frac{\mu}{E_{\text{out}}}\right)\sum_{(ij)}\boldsymbol{T}_i\cdot\boldsymbol{T}_j\int[d^2\Omega_q]\,W_{ij}^q\Theta_{\text{out}}(n_q)$$

So why not just use the $\overline{\text{XS}}$ scheme?

The angular integral of the $\overline{\text{XS}}$ scheme **anomalous dimension** reads:

$$\mathcal{S}_{ij}^{\overline{\text{XS}}} = \int[d\Omega_q]\,W_{ij}^q\,\theta_{\text{out}}(n_q) = \left(\frac{e^{\gamma_E}}{\pi}\right)^{\epsilon}\int\frac{d^{d-2}\Omega_q}{2(2\pi)^{d-3}}\,(n_{vq})^{2\epsilon}\,W_{ij}^q\,\theta_{\text{out}}(n_q)$$

An explicit reference vector v is needed for Lorentz covariance (LC). LC is not “manifest”.

The $\overline{\mathrm{XS}}$-Scheme Anomalous Dimension

Two-Loop (2112.02108) (for Lepton Colliders)

$$\begin{aligned}\boldsymbol{\Gamma}^{(2)} = {} & K_{ijk;qr}\left(\boldsymbol{T}_{ijk}^{\alpha\beta;\tilde{\alpha}\tilde{\beta}} - 2\boldsymbol{T}_{ijk}^{\alpha;\tilde{\alpha}} + \boldsymbol{T}_{ijk}\right) + \text{h.c.} \\ & - 2K_{ij;qr}^{A}\boldsymbol{D}_{ij}^{A,\alpha\beta;\tilde{\alpha}\tilde{\beta}} - 2n_F K_{ij;qr}^{F}\boldsymbol{D}_{ij}^{F,\alpha\beta;\tilde{\alpha}\tilde{\beta}} - 2\, n_S K_{ij;qr}^{S}\boldsymbol{D}_{ij}^{S,\alpha\beta;\tilde{\alpha}\tilde{\beta}} \\ & + 2(C_A K_{ij;qr}^{A} + n_F T_F K_{ij;qr}^{F} + n_S T_S K_{ij;qr}^{S})\boldsymbol{D}_{ij}^{\alpha;\tilde{\alpha}} \\ & + K_{ij;q}\left(2\boldsymbol{D}_{ij}^{\alpha;\tilde{\alpha}} - \boldsymbol{D}_{ij} - \overline{\boldsymbol{D}}_{ij}\right) \\ & + i\pi I_{ijk;q}\left(\boldsymbol{T}_{ijk}^{\alpha;\tilde{\alpha}} - \overline{\boldsymbol{T}}_{ijk}^{\alpha;\tilde{\alpha}}\right) - i\pi I_{ij}\left(\boldsymbol{D}_{ij} - \overline{\boldsymbol{D}}_{ij}\right)\,,\end{aligned}$$

=0

The three-particle term contains an explicit v-dependence:

$$K_{ijk;qr} = 8\left(W_{ik}^{q}W_{jk}^{r} - W_{ik}^{q}W_{jq}^{r} - W_{ir}^{q}W_{jk}^{r} + W_{ij}^{q}W_{jq}^{r}\right)\ln\left(\frac{n_{kq}\, n_{vr}}{n_{kr}\, n_{vq}}\right)$$

Scheme Change To Lorentz Scheme

The one-loop anomalous dimension in the Lorentz scheme (**LS**) reads

Modified dipole (Cancels explicit v -dependence in $[d\Omega_q]$)

$$
\begin{aligned}
\overline{\boldsymbol{R}}_m =& -4\sum_{(ij)} \boldsymbol{T}^a_{i,L}\boldsymbol{T}^{\tilde{a}}_{j,R} W^q_{ij}\left(2W^q_{ij}n^2_{vq}\right)^{-\epsilon}\theta_{\text{in}}\left(n_q\right) \\
\overline{\boldsymbol{V}}_m =& 2\sum_{(ij)}\left(\boldsymbol{T}_{i,L}\cdot\boldsymbol{T}_{j,L}+\boldsymbol{T}_{i,R}\cdot\boldsymbol{T}_{j,R}\right)\int[d\Omega_q]W^q_{ij}\left(2W^q_{ij}n^2_{vq}\right)^{-\epsilon} \\
& -i\pi\sum_{(ij)}\frac{1}{2}\left[\boldsymbol{T}_{i,L}\cdot\boldsymbol{T}_{j,L}-\boldsymbol{T}_{i,R}\cdot\boldsymbol{T}_{j,R}\right]\left(\Pi_{ij}-\epsilon\ln(n_{ij})\right)\gamma_0^{\text{cusp}}
\end{aligned}
$$

Modified phase (non-vanishing even in lepton collider processes)

Scheme Change To Lorentz Scheme

For lepton colliders, this leads to the two-loop anomalous dimension

$$
\begin{aligned}
\boldsymbol{\Gamma}^{(2)} = {} & K_{ijk;qr}\left(\boldsymbol{T}_{ijk}^{\alpha\beta;\tilde{\alpha}\tilde{\beta}} - 2\boldsymbol{T}_{ijk}^{\alpha;\tilde{\alpha}} + \boldsymbol{T}_{ijk}\right) + \text{h.c.} \\
& - 2K_{ij;qr}^{A}\boldsymbol{D}_{ij}^{A,\alpha\beta;\tilde{\alpha}\tilde{\beta}} - 2n_F K_{ij;qr}^{F}\boldsymbol{D}_{ij}^{F,\alpha\beta;\tilde{\alpha}\tilde{\beta}} - 2\,n_S K_{ij;qr}^{S}\boldsymbol{D}_{ij}^{S,\alpha\beta;\tilde{\alpha}\tilde{\beta}} \\
& + 2(C_A K_{ij;qr}^{A} + n_F T_F K_{ij;qr}^{F} + n_S T_S K_{ij;qr}^{S})\boldsymbol{D}_{ij}^{\alpha;\tilde{\alpha}} \\
& + K_{ij;q}\left(2\boldsymbol{D}_{ij}^{\alpha;\tilde{\alpha}} - \boldsymbol{D}_{ij} - \overline{\boldsymbol{D}}_{ij}\right) \\
& + i\pi I_{ijk;q}\left(\boldsymbol{T}_{ijk}^{\alpha;\tilde{\alpha}} - \overline{\boldsymbol{T}}_{ijk}^{\alpha;\tilde{\alpha}}\right) - i\pi I_{ij}\left(\boldsymbol{D}_{ij} - \overline{\boldsymbol{D}}_{ij}\right),
\end{aligned}
$$

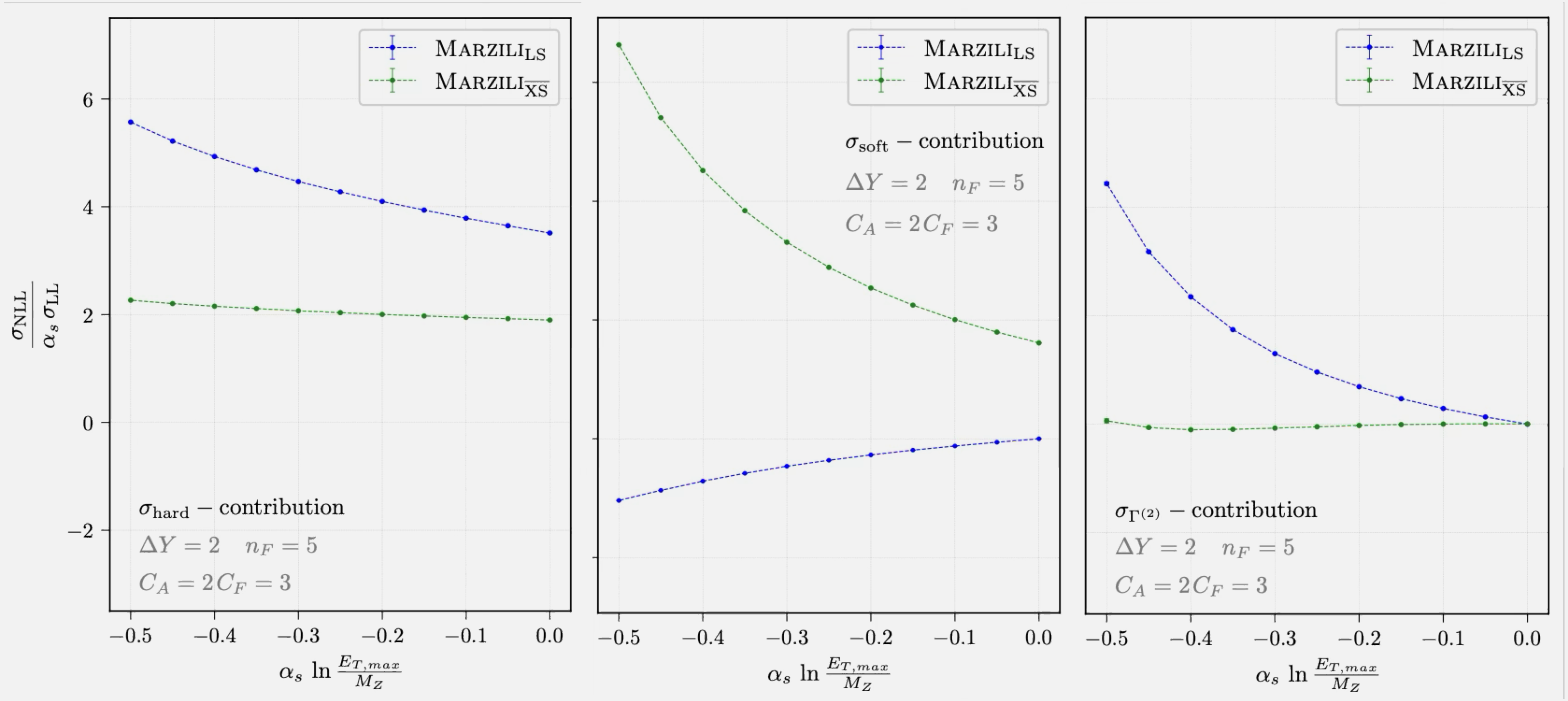
=0

The 3-particle term now reads

$$
\begin{aligned}
\overline{K}_{ijk;qr} = \frac{4}{n_{iq}n_{jr}}\Bigg(& \frac{n_{ij}}{n_{qr}}\ln\left(\frac{n_{ir}n_{jr}n_{kq}^2}{n_{iq}n_{jq}n_{kr}^2}\right) + \frac{n_{ik}n_{jk}}{n_{kq}n_{kr}}\ln\left(\frac{n_{ik}n_{jr}n_{kq}}{n_{iq}n_{jk}n_{kr}}\right) - \frac{n_{iq}n_{jk}}{n_{kq}n_{qr}}\ln\left(\frac{n_{jk}n_{qr}}{n_{jq}n_{kr}}\right) \\
& - \frac{n_{ik}n_{jq}}{n_{kq}n_{qr}}\ln\left(\frac{n_{ik}n_{jr}n_{kq}n_{qr}}{n_{iq}n_{jq}n_{kr}^2}\right) + \frac{n_{ir}n_{jk}}{n_{kr}n_{qr}}\ln\left(\frac{n_{iq}n_{jk}n_{kr}n_{qr}}{n_{ir}n_{jr}n_{kq}^2}\right) + \frac{n_{ik}n_{jr}}{n_{kr}n_{qr}}\ln\left(\frac{n_{ik}n_{qr}}{n_{ir}n_{kq}}\right)\Bigg)
\end{aligned}
$$

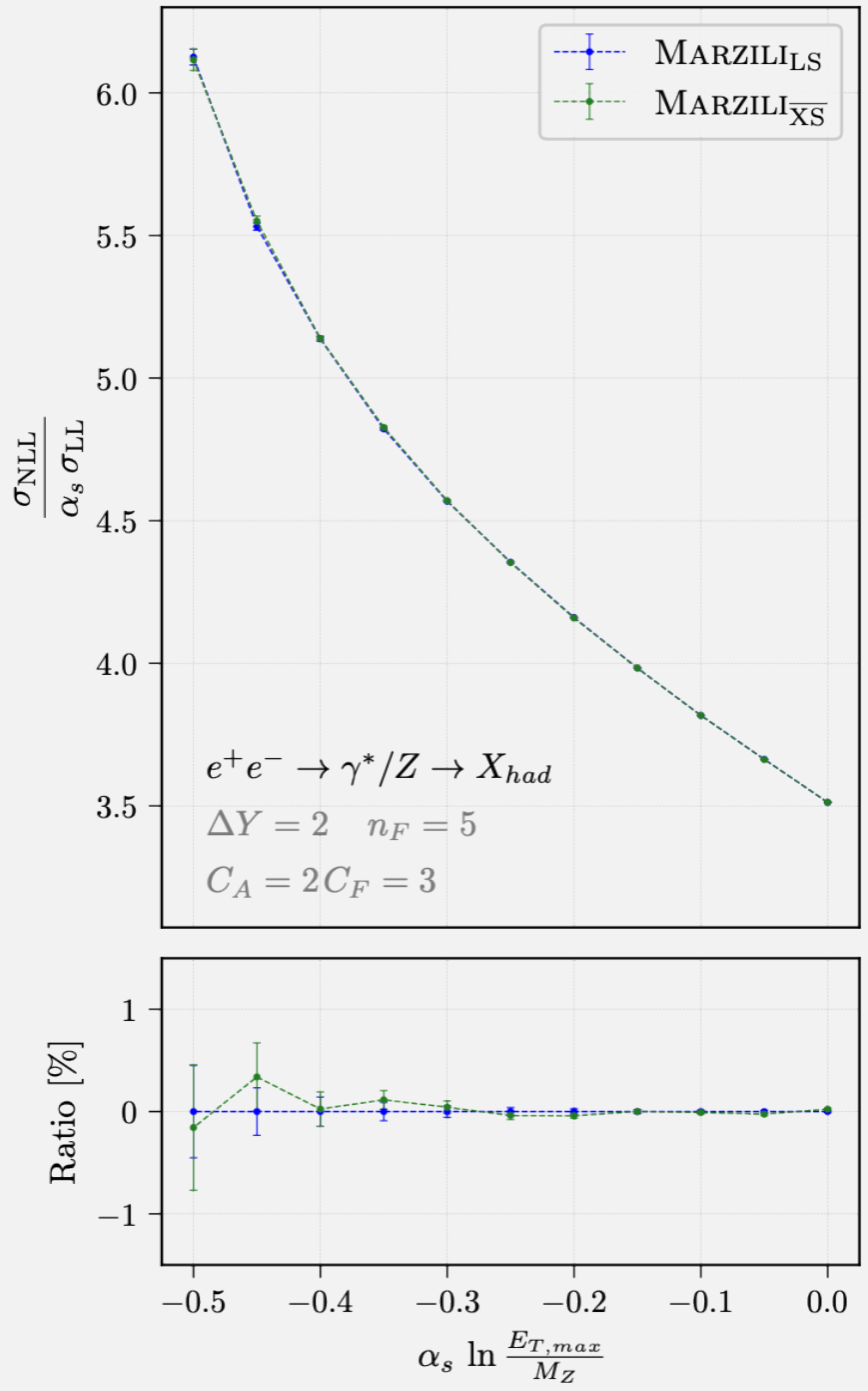
The explicit v-dependence has vanished! The LS-scheme anomalous dimension is manifestly Lorentz covariant!

Marzili $\overline{\mathbf{XS}}$ vs. $\mathbf{LS}$ Scheme



- Plots show NLL correction, normalized to $\alpha_s \sigma_{LL}$ as in PanScales '23. Limit $N_C \to \infty$.
- Individual parts strongly differ between schemes!

Marzili $\overline{\textbf{XS}}$ vs. **LS** Scheme



- Individual terms strongly differ...
- ... but the sum is scheme independent and agrees!
- Strong check on implementation and numerics!

Part 2: Including Clustering Effects

Why We Need Energy Fractions: LL

The soft function for non-global logs in jet clustering cross sections reads

$$\boldsymbol{\mathcal{S}}_m(\{\underline{n}\},\{\underline{z}\},Q_0) = \sum_X\!\!\!\!\!\!\int \langle 0|\boldsymbol{\mathcal{S}}_1^\dagger(n_1)\ldots\boldsymbol{\mathcal{S}}_m^\dagger(n_m)|X\rangle\langle X|\boldsymbol{\mathcal{S}}_1(n_1)\ldots\boldsymbol{\mathcal{S}}_m(n_m)|0\rangle\theta(Q_0-2E_{\text{out}})$$

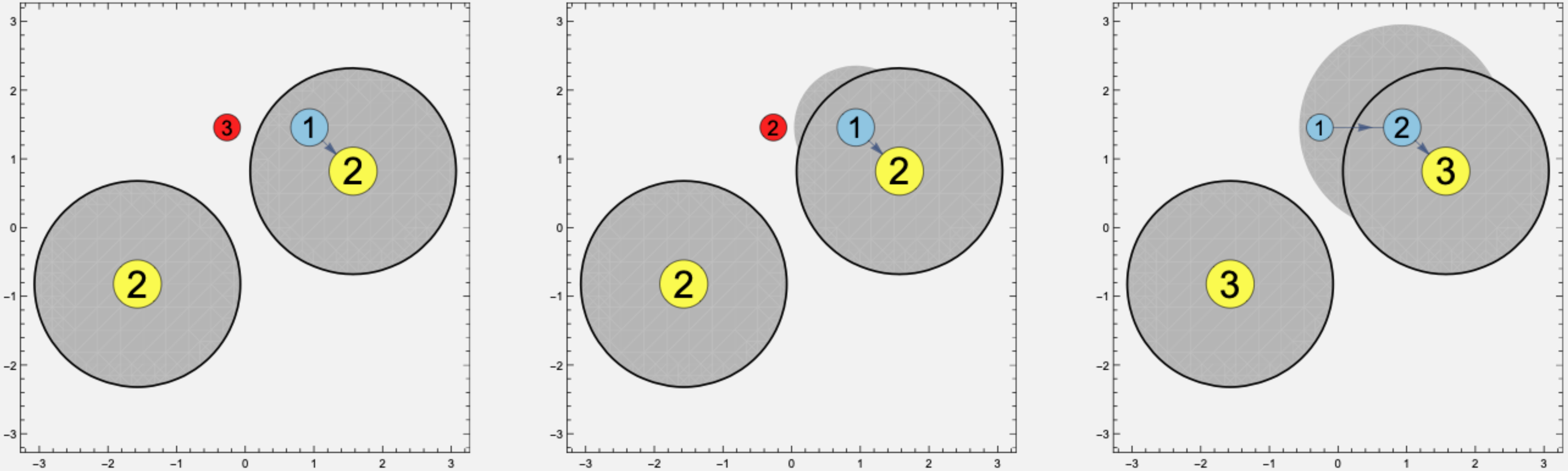
E_{out} depends on the energy fractions of the hard partons! They are needed to determine the clustering sequence.

At LL, generated emissions in the RG evolution are strongly ordered in energy. This manifests in the one-loop anomalous dimension as

$$\boldsymbol{R}_m^{cl} = \boldsymbol{R}_m\delta(z)\mathrm{d}z$$
$$\boldsymbol{V}_m^{cl} = \boldsymbol{V}_m$$

It becomes harder to determine wheter an emission is vetoed or not. Everything else works as in the fixed cone cross sections.

Example Of LL Clustering Effects



(a) Anti-k_t. (b) C/A. (c) k_t.

What we found in 2309.17355:

1. Jet clustering logarithms generally depend on the clustering algorithm choice already at **2 loops**.
2. The primary hard jets grow in C/A and k_t-algorithms but stay the same in Anti-k_t.
3. Even with Anti-k_t , it is possible to get clustering logarithms if the veto region is a non-trivial function of the secondary jets!

Why We Need Energy Fractions: NLL

At NLL, two emissions with momenta, q and r can be generated at the same energy scale!

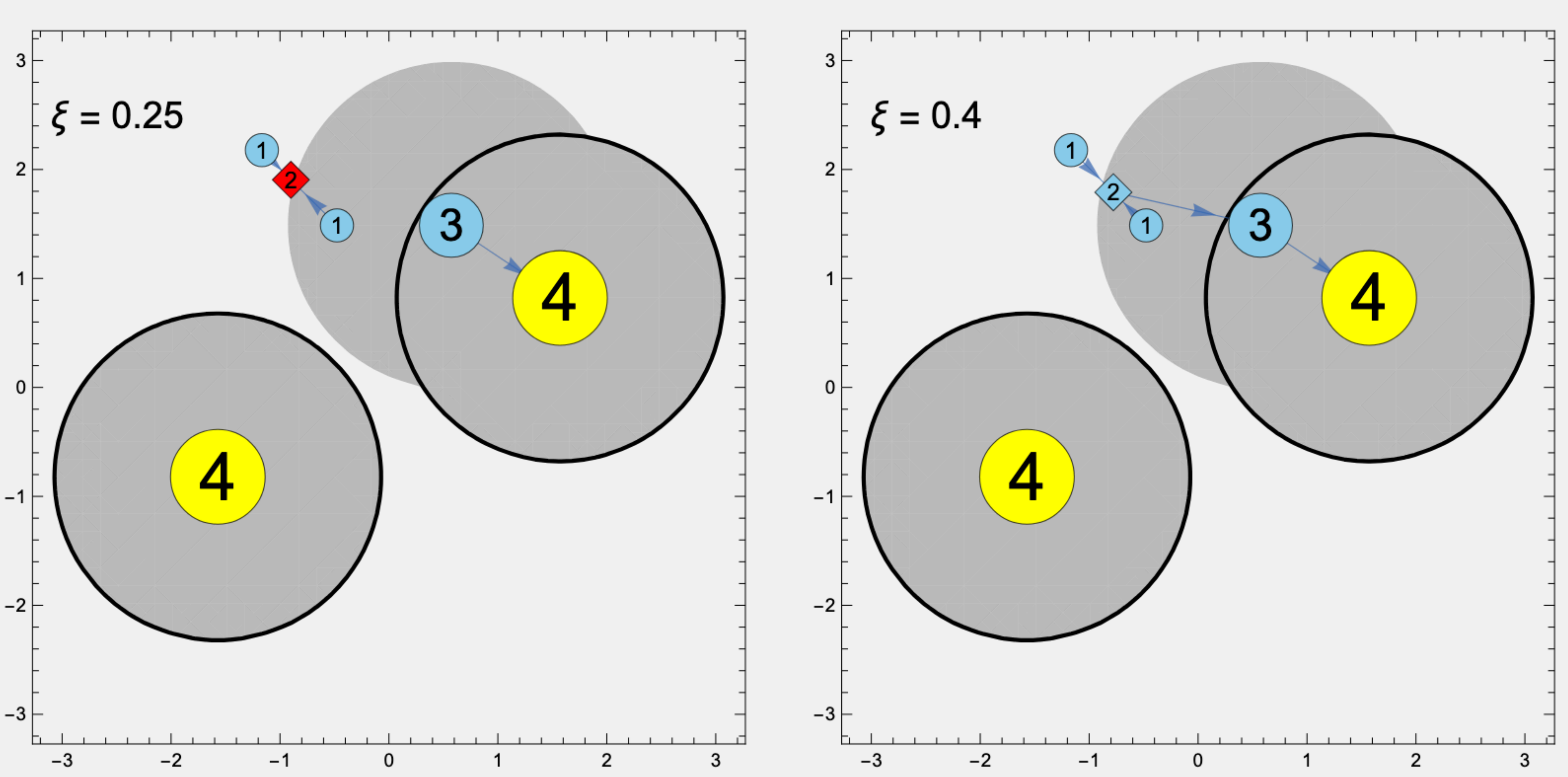
The energy dependence of the double emission anomalous dimension can be parametrized using

$$z = \frac{E_q+E_r}{E_m} \quad \xi = \frac{E_q}{E_q+E_r}$$

3rd softest emission

d_m^{cl} Is clrealy proportional to $\delta(z)$. But the ξ -dependence can be highly non-trivial!

Why We Need Energy Fractions: NLL



Two configurations with the same particle directions can be vetoed or not depending on the value of ξ!

The Clustering Two-Loop Anomalous Dimension

The double-virtual and real-virtual anomalous dimensions can be recycled from the fixed cone result:

$$\boldsymbol{r}_m^{cl} = \boldsymbol{r}_m \delta(z)\mathrm{d}z$$

$$\boldsymbol{v}_m^{cl} = \boldsymbol{v}_m$$

The double-emission anomalous dimension reads

$$\begin{aligned}\boldsymbol{d}_m^{cl} = \delta(z)\mathrm{d}z \Big[& \mathrm{d}K_{ijkl;qr} \left(\boldsymbol{T}_{i,L}^{\alpha}\boldsymbol{T}_{j,L}^{\beta}\right)_+ \left(\boldsymbol{T}_{k,R}^{\tilde{\alpha}}\boldsymbol{T}_{R,l}^{\tilde{\beta}}\right)_+ \\ & + \mathrm{d}K_{ijk;qr} \left(i f^{\tilde{\alpha}\tilde{\beta}c} \left(\boldsymbol{T}_{i,L}^{\alpha}\boldsymbol{T}_{j,L}^{\beta}\right)_+ \boldsymbol{T}_{k,R}^{c} - i f^{\alpha\beta c} \left(\boldsymbol{T}_{k,L}^{c} \left(\boldsymbol{T}_{i,R}^{\tilde{\alpha}}\boldsymbol{T}_{j,R}^{\tilde{\beta}}\right)_+\right)\right) \\ & - 2\left(\mathrm{d}K_{ij;qr}^{A} + n_S \mathrm{d}K_{ij;qr}^{S}\right) \boldsymbol{T}_{i,L}^{c}\boldsymbol{T}_{j,R}^{d} f^{\alpha\beta c} f^{\tilde{\alpha}\tilde{\beta}d} - 2n_F \mathrm{d}K_{ij;qr}^{F} \boldsymbol{T}_{i,L}^{c}\boldsymbol{T}_{j,R}^{d} t^{c,\alpha}{}_{\beta} t^{d,\tilde{\beta}}{}_{\tilde{\alpha}} \Big]\end{aligned}$$

Note that there is a four-particle term!

The Clustering Two-Loop Anomalous Dimension

The emission kernels depend on ξ . In $\overline{\mathrm{XS}}$-scheme they can be written in terms of eikonal currents:

$$\tilde{J}_i^{\mu}(n_q) = E_q J_i^{\mu}(q)\,, \quad \tilde{J}_i^{\mu\nu}(n_q, n_r, \xi) = E_q E_r J_i^{\mu\nu}(q, r)$$

$$\mathrm{d}K_{ijkl;qr} = 8\frac{\mathrm{d}\xi}{(\xi\bar{\xi})_+}\tilde{J}_{i,\mu}\tilde{J}_i^{\mu}\tilde{J}_{j,\nu}\tilde{J}_l^{\nu}$$

$$\mathrm{d}K_{ijk;qr} = -8\frac{\mathrm{d}\xi}{(\xi\bar{\xi})_+}\tilde{J}_k^{\mu\nu}\tilde{J}_{i,\mu}\tilde{J}_{j,\nu}$$

$$\mathrm{d}K_{ij;qr}^{A} = -4\frac{\mathrm{d}\xi}{(\xi\bar{\xi})_+}\left(\tilde{J}_i^{\mu\nu}\tilde{J}_{j,\mu\nu} - \frac{1}{2}\tilde{J}_i^{\mu\nu}\tilde{J}_{i,\mu\nu} - \frac{1}{2}\tilde{J}_j^{\mu\nu}\tilde{J}_{j,\mu\nu}\right)\Bigg|_{\epsilon=0}$$

$$\mathrm{d}K_{ij;qr}^{F} = -8\frac{\mathrm{d}\xi}{\xi\bar{\xi}}\frac{1}{(n_q\cdot n_r)^2}\left(-g_{\mu\nu}q\cdot r + q_\mu r_\nu + r_\mu q_\nu\right)\left(J_i^{\mu}J_j^{\nu} - \frac{1}{2}J_i^{\mu}J_i^{\nu} - \frac{1}{2}J_j^{\mu}J_j^{\nu}\right)$$

Again, no manifest Lorentz covariance!

e.g.

$$\mathrm{d}K_{ijk;qr} = \frac{1}{4}\mathrm{d}\xi\left(\delta(\xi)+\delta(\bar{\xi})\right)K_{ijk;qr} - 4\mathrm{d}\xi(\delta(\xi)-\delta(\bar{\xi}))W_{ir}^{q}W_{jk}^{r}\ln\left(\frac{n_{kq}n_{vr}}{n_{kr}n_{vq}}\right) + 4\mathrm{d}\xi\left(\frac{n_{kr}}{\xi\left(\xi n_{kq}+\bar{\xi}n_{kr}\right)}\right)_+\left(2W_{ir}^{q}W_{jk}^{r} - W_{ik}^{q}W_{jk}^{r} - W_{ij}^{r}W_{ir}^{q}\right) - \left((i,q,\xi)\leftrightarrow(j,r,\bar{\xi})\right)$$

The blue pieces integrate to zero! They constitute an effect unique to clustering logarithms!

Summary

- Subtleties appear for non-global logarithms in the $\overline{\mathrm{MS}}$-scheme.
- The $\overline{\mathrm{XS}}$ scheme is intuitive; however, it is not manifestly Lorentz covariant.
- We analyzed scheme changes in general.
- The $\overline{\mathrm{LS}}$-scheme is manifestly Lorentz covariant.
- Clustering logarithms requires explicit energy fraction dependence.
- We provide, for the first time, the anomalous dimensions for clustering logarithms at two loops and in full colour.

Thank You!

Backup Slides

One-Loop Soft function

$$\frac{\alpha_s}{4\pi}\boldsymbol{S}_m^{(1)}(\{\underline{n}\},Q_0) = -g_s^2\sum_{(ij)} \boldsymbol{T}_{i,L}\cdot\boldsymbol{T}_{j,R}\,\tilde{\mu}^{4-d}\int\frac{d^dq}{(2\pi)^{d-1}}W_{ij}^q\delta(q^2)\theta(q^0)\,\theta(E_{\rm out}-v\cdot q)\Theta_{\rm out}(n_q)$$

- Performing the energy integral, we find

$$\boldsymbol{\mathcal{S}}_m^{(1)}(\{\underline{n}\},E_{\rm out}) = \frac{2}{\epsilon}\left(\frac{\mu}{E_{\rm out}}\right)^{2\epsilon}\sum_{(ij)}\boldsymbol{T}_i\cdot\boldsymbol{T}_j\,\mathcal{S}_{ij}$$

where $\mathcal{S}_{ij} = \int[d\Omega_q]\,W_{ij}^q\,\Theta_{\rm out}(n_q)$, and $\int[d\Omega_q] \equiv \left(\frac{e^{\gamma_E}}{\pi}\right)^{\epsilon}\int\frac{d^{d-2}\Omega_q}{2(2\pi)^{d-3}}\,(n_{vq})^{2\epsilon}$

- The angular integral can be expanded in ϵ: $\mathcal{S}_{ij} = \mathcal{S}_{ij}^{[0]} + \epsilon\,\mathcal{S}_{ij}^{[1]} + \dots$
- To expand the angular integral, we need to know which angles $\theta_{\rm out}$ can depend on. Here we assume $\theta_{\rm out} = \theta_{\rm out}(v\cdot n_q, n_i\cdot n_q, n_j\cdot n_q, n_\perp\cdot n_q)$, where $n_\perp$ is any direction linearly independent from n_i and n_j

One-Loop Soft function

$$\mathcal{S}_{ij} = \int [d\Omega_q]\, W_{ij}^q\, \Theta_{\rm out}(n_q)\,, \qquad \theta_{\rm out} = \theta_{\rm out}(v \cdot n_q, n_i \cdot n_q, n_j \cdot n_q, n_\perp \cdot n_q)$$

This integral is conveniently studied in the dipole rest frame. We can then parametrize

$$n_q \equiv n_q(\theta,\phi,\chi) = (1, \cos\theta, \sin\theta\cos\phi, \sin\theta\cos\phi\cos\chi, \sin\theta\sin\chi\,\hat{n}_{d-4})$$

The angular phase space can then be expanded as

Requires +-prescription

$$\int [d\Omega_q] = \frac{e^{\epsilon\gamma_E}\Omega_{d-4}}{(4\pi)^{1-2\epsilon}} \int_0^\pi d\theta\,(\sin\theta)^{1-2\epsilon} \int_{-\pi}^{\pi} d\phi\,(\sin\phi)^{-2\epsilon} \int_0^{\frac{\pi}{2}} d\chi\,(\sin\chi)^{-1-2\epsilon}\,(n_{vq})^{2\epsilon}$$

The integration domain $\chi \in [\pi/2, \pi]$ is captured by extending the ϕ integration to negative values

We now find that $$\mathcal{S}_{ij}^{[0]} = \frac{1}{4\pi}\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi \frac{2}{\sin^2\theta}\theta_{\rm out}\left(\tilde{n}_q\right)$$

Where $\tilde{n}_q \equiv n_q(\theta,\phi,0) = \left(1, \cos\theta, \sin\theta\cos\phi, \sin\theta\cos\phi, \vec{0}_{d-4}\right)$

One-Loop Soft function: Finite Piece

We now find that $\mathcal{S}_{ij}^{[0]} = \frac{1}{4\pi}\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi \frac{2}{\sin^2\theta}\theta_{\text{out}}\left(\tilde{n}_q\right)$ $\quad \tilde{n}_q \equiv n_q(\theta,\phi,0) = \left(1, \cos\theta, \sin\theta\cos\phi, \sin\theta\cos\phi, \vec{0}_{d-4}\right)$

$\mathcal{S}_{ij}^{[0]}$ defines the $\overline{\mathrm{MS}}$ counter term! The $\overline{\mathrm{MS}}$-renormalized soft function would then read

$$\boldsymbol{\mathcal{S}}^{\overline{\mathrm{MS}}(1)} = \sum_{(ij)} \boldsymbol{T}_i \cdot \boldsymbol{T}_j \left[4\ln\left(\frac{\mu}{Q_0}\right)\mathcal{S}_{ij}^{[0]} + \mathcal{S}_{ij}^{(1)}\right]$$

where

$$\mathcal{S}_{ij}^{[1]} = \frac{1}{4\pi}\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi \frac{2}{\sin^2\theta}\int_0^{\frac{\pi}{2}} d\chi \left\{ \frac{1}{\sin\chi}\left[\theta_{\text{out}}\left(\tilde{n}_q\right) - \theta_{\text{out}}\left(n_q\right)\right] + \frac{2}{\pi}\theta_{\text{out}}\left(\tilde{n}_q\right)\left[2\ln(n_{vq}) - \ln\left(\sin^2\theta\right) - \ln\left(\sin^2\phi\right)\right]\right\}$$

Thus, the $\overline{\mathrm{MS}}$-renormalized soft function is very complicated! It depends on the extra angle χ and sees the full $4-2\epsilon$ dimensional veto region!

The $\overline{\mathrm{MS}}$-scheme requires sampling hard and soft functions with extra angles! The extra angles only drop out after cancellations with the two-loop anomalous dimension.

Anticipating Some Questions

$$\boldsymbol{\mathcal{S}}^{\overline{\text{MS}}(1)} = \sum_{(ij)} \boldsymbol{T}_i \cdot \boldsymbol{T}_j \left[4\ln\left(\frac{\mu}{Q_0}\right) \mathcal{S}_{ij}^{[0]} + \mathcal{S}_{ij}^{(1)} \right]$$

$$\tilde{n}_q \equiv n_q(\theta, \phi, 0) = \left(1, \cos\theta, \sin\theta\cos\phi, \sin\theta\cos\phi, \vec{0}_{d-4}\right)$$

$$\mathcal{S}_{ij}^{[1]} = \frac{1}{4\pi}\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi \frac{2}{\sin^2\theta} \int_0^{\frac{\pi}{2}} d\chi \left\{ \frac{1}{\sin\chi}\left[\theta_{\text{out}}(\tilde{n}_q) - \theta_{\text{out}}(n_q)\right] + \frac{2}{\pi}\theta_{\text{out}}(\tilde{n}_q)\left[2\ln(n_{vq}) - \ln\left(\sin^2\theta\right) - \ln\left(\sin^2\phi\right)\right]\right\}$$

- Q: Can't I rescue $\overline{\text{MS}}$ by simply defining the d-dimensional veto region as $\theta_{\text{out}}(n_q) \equiv \theta_{\text{out}}(\tilde{n}_q)$.

  A: No! $\tilde{n}_q$ depends on the radiating dipole!

- Q: Isn't even the bare soft function not well defined, because I can define the d-dimensional veto region however I want?

  A: In a sense, yes. The physical result should not depend on how we extend the veto region to d-dimensions. Thus, schemes where $\theta_{\text{out}}(n_q)$ drops out completely are favourable.

- Q: Why is this issue with $\overline{\text{MS}}$ only present for non-global observables?

  A: Assume we veto on $E_q f(\theta, \phi, \chi)$ instead. This would produce a factor $f^{2\epsilon}(\theta, \phi, \chi) \Rightarrow \chi$ enters one order in ϵ later!

The $\overline{\text{XS}}$-Scheme Anomalous Dimension

The anomalous dimension for non-global logarithms has a matrix structure:

$$\mathbf{\Gamma}^{(1)} = \begin{pmatrix} \boldsymbol{V}_2 & \boldsymbol{R}_2 & 0 & 0 & \dots \\ 0 & \boldsymbol{V}_3 & \boldsymbol{R}_3 & 0 & \cdots \\ 0 & 0 & \boldsymbol{V}_4 & \boldsymbol{R}_4 & \cdots \\ 0 & 0 & 0 & \boldsymbol{V}_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \mathbf{\Gamma}^{(2)} = \begin{pmatrix} \boldsymbol{v}_2 & \boldsymbol{r}_2 & \boldsymbol{d}_2 & 0 & \dots \\ 0 & \boldsymbol{v}_3 & \boldsymbol{r}_3 & \boldsymbol{d}_2 & \cdots \\ 0 & 0 & \boldsymbol{v}_4 & \boldsymbol{r}_4 & \cdots \\ 0 & 0 & 0 & \boldsymbol{v}_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

One-Loop

$$\begin{aligned} \boldsymbol{R}_m =& -4 \sum_{(ij)} \boldsymbol{T}_{i,L}^{\alpha} \boldsymbol{T}_{j,R}^{\tilde{\alpha}} W_{ij}^{q}\, \theta_{\text{in}}(n_q) \\ \boldsymbol{V}_m =& 2 \sum_{(ij)} (\boldsymbol{T}_{i,L} \cdot \boldsymbol{T}_{j,L} + \boldsymbol{T}_{i,R} \cdot \boldsymbol{T}_{j,R}) \int [d\Omega_q]\, W_{ij}^{q} \\ & - i\pi \sum_{(ij)} \frac{1}{2} \left[\boldsymbol{T}_{i,L} \cdot \boldsymbol{T}_{j,L} - \boldsymbol{T}_{i,R} \cdot \boldsymbol{T}_{j,R}\right] \Pi_{ij} \gamma_0^{\text{cusp}} \end{aligned}$$

Introducing the short hand notation

$$\boldsymbol{D}_{ij}^{\alpha\tilde{\alpha}} = \boldsymbol{T}_{i,L}^{\alpha}\, \boldsymbol{T}_{j,R}^{\tilde{\alpha}} = \overline{\boldsymbol{D}}_{ji}^{\alpha\tilde{\alpha}}\,, \qquad \boldsymbol{D}_{ij} = \boldsymbol{T}_{i,L} \cdot \boldsymbol{T}_{j,L}\,, \qquad \overline{\boldsymbol{D}}_{ij} = \boldsymbol{T}_{i,R} \cdot \boldsymbol{T}_{j,R}$$

We can write this as

$$\mathbf{\Gamma}^{(1)} = -4 W_{ij}^{q} \left(\boldsymbol{D}_{ij}^{\alpha\tilde{\alpha}} - \frac{1}{2} \boldsymbol{D}_{ij} - \frac{1}{2} \overline{\boldsymbol{D}}_{ij} \right)$$

Scheme Change

To change from the $\overline{\mathbf{XS}}$-scheme to any other scheme $\mathbf{RS}$, one calculates

$$\boxed{\Delta\mathbf{\Gamma}^{\mathrm{RS}(2)} = \mathbf{\Gamma}^{\mathrm{RS}(2)} - \mathbf{\Gamma}^{(2)} = \left[\delta\mathbf{\Gamma}^{\mathrm{RS}(1)}, \mathbf{\Gamma}^{(1)}\right] - 2\beta_0\, \delta\mathbf{\Gamma}^{\mathrm{RS}(1)}}$$

where $\delta\mathbf{\Gamma}^{\mathrm{RS}(1)} = -\lim_{\epsilon\to 0}\frac{1}{2\epsilon}\left(\mathbf{\Gamma}^{\mathrm{RS}(1)} - \mathbf{\Gamma}^{(1)}\right)$

$$\begin{aligned}
\delta\boldsymbol{R}_m^{\mathrm{RS}} &= -4\sum_{i,j=1}^{m} \boldsymbol{T}_{i,L}^{\alpha}\boldsymbol{T}_{j,R}^{\tilde{\alpha}}\,\delta W_{ij}^{q}\,\theta_{\mathrm{in}}(n_q) \\
\delta\boldsymbol{V}_m^{\mathrm{RS}} &= 2\sum_{i,j=1}^{m} \left(\boldsymbol{T}_{i,L}\cdot\boldsymbol{T}_{j,L} + \boldsymbol{T}_{i,R}\cdot\boldsymbol{T}_{j,R}\right)\int[d\Omega_q]\,\delta W_{ij}^{q} \\
&\quad - i\pi\sum_{i,j=1}^{m}\frac{1}{2}\left[\boldsymbol{T}_{i,L}\cdot\boldsymbol{T}_{j,L} - \boldsymbol{T}_{i,R}\cdot\boldsymbol{T}_{j,R}\right]\delta\Pi_{ij}\gamma_0^{\mathrm{cusp}}
\end{aligned}$$

Explicit Scheme Change Formula

$$\begin{aligned}\Delta\boldsymbol{\Gamma}^{\mathrm{RS}(2)} =& \delta K_{ijk;qr}\left(\boldsymbol{T}^{\alpha\beta;\tilde\alpha\tilde\beta}_{ijk} - 2\boldsymbol{T}^{\alpha;\tilde\alpha}_{ijk} + \boldsymbol{T}_{ijk} + \mathrm{h.c.}\right) + \delta K^{v}_{ijk}\left(\boldsymbol{T}_{ijk} + \bar{\boldsymbol{T}}_{ijk}\right)\\ &- 2\delta K^{A}_{ij;qr}\left(\boldsymbol{D}^{A,\alpha\beta;\tilde\alpha\tilde\beta}_{ij} - C_A\boldsymbol{D}^{\alpha;\tilde\alpha}_{ij}\right) - 2\beta_0\delta W^{q}_{ij}\left(2\boldsymbol{D}^{\alpha;\tilde\alpha}_{ij} - (\boldsymbol{D}_{ij} + \overline{\boldsymbol{D}}_{ij})\right)\\ &+ i\pi\delta I_{ijk;q}\left(\boldsymbol{T}^{\alpha;\tilde\alpha}_{ijk} - \bar{\boldsymbol{T}}^{\alpha;\tilde\alpha}_{ijk}\right) + i\pi\delta I^{v}_{ijk}\left(\boldsymbol{T}_{ijk} - \bar{\boldsymbol{T}}_{ijk}\right)\\ &- i\pi(C_A\delta I_{ij;q} + 2\beta_0\delta\Pi_{ij})\left(\boldsymbol{D}_{ij} - \overline{\boldsymbol{D}}_{ij}\right)\end{aligned}$$

Extra terms for hadron colliders!

Color Structures

$$\boldsymbol{D}_{ij}^{\alpha\tilde{\alpha}} = \boldsymbol{T}_{i,L}^{\alpha}\,\boldsymbol{T}_{j,R}^{\tilde{\alpha}} = \overline{\boldsymbol{D}}_{ji}^{\alpha\tilde{\alpha}}\,, \qquad \boldsymbol{D}_{ij} = \boldsymbol{T}_{i,L}\cdot\boldsymbol{T}_{j,L}\,, \qquad \overline{\boldsymbol{D}}_{ij} = \boldsymbol{T}_{i,R}\cdot\boldsymbol{T}_{j,R}$$

$$\boldsymbol{D}_{ij}^{A,\alpha\beta;\tilde{\alpha}\tilde{\beta}} = \boldsymbol{T}_{i,L}^{c}\boldsymbol{T}_{j,R}^{d} f^{\alpha\beta c} f^{\tilde{\alpha}\tilde{\beta}d}\,, \qquad \boldsymbol{D}_{ij}^{F,\alpha\beta;\tilde{\alpha}\tilde{\beta}} = \boldsymbol{T}_{i,L}^{c}\boldsymbol{T}_{j,R}^{d} t^{c,\alpha}{}_{\beta} t^{d,\tilde{\beta}}{}_{\tilde{\alpha}}$$

$$\boldsymbol{T}_{ijk}^{\alpha\beta;\tilde{\alpha}\tilde{\beta}} = i f^{\tilde{\alpha}\tilde{\beta}c}\left(\boldsymbol{T}_{i,L}^{\alpha}\boldsymbol{T}_{j,L}^{\beta}\right)_{+}\boldsymbol{T}_{k,R}^{c} \qquad \overline{\boldsymbol{T}}_{ijk}^{\alpha\beta;\tilde{\alpha}\tilde{\beta}} = -i f^{\alpha\beta c}\boldsymbol{T}_{k,L}^{c}\left(\boldsymbol{T}_{j,R}^{\tilde{\beta}}\boldsymbol{T}_{i,R}^{\tilde{\alpha}}\right)_{+}$$

$$\boldsymbol{T}_{ijk}^{\alpha;\tilde{\alpha}} = i f^{\tilde{\alpha}bc}\boldsymbol{T}_{i,L}^{\alpha}\left(\boldsymbol{T}_{k,R}^{c}\boldsymbol{T}_{j,R}^{b}\right)_{+} \qquad \overline{\boldsymbol{T}}_{ijk}^{\alpha;\tilde{\alpha}} = -i f^{\alpha bc}\left(\boldsymbol{T}_{j,L}^{b}\boldsymbol{T}_{k,L}^{c}\right)_{+}\boldsymbol{T}_{i,R}^{\tilde{\alpha}}$$

$$\boldsymbol{T}_{ijk} = i f^{abc}\left(\boldsymbol{T}_{k,R}^{c}\boldsymbol{T}_{j,R}^{b}\boldsymbol{T}_{i,R}^{a}\right)_{+} \qquad \overline{\boldsymbol{T}}_{ijk} = -i f^{abc}\left(\boldsymbol{T}_{i,L}^{a}\boldsymbol{T}_{j,L}^{b}\boldsymbol{T}_{k,L}^{c}\right)_{+}$$