# $B_c \to \eta_c$ form factors at large recoil: SCET analysis and a three-loop check

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# SCET at next-to-leading power

- ▶ Significant interest in SCET at next-to-leading power in recent years, e.g.:

  - ▶ Threshold resummation [Beneke et al. 18-23]
  - ▶ $H \to b\bar{b} \to \gamma\gamma$ and $gg \to b\bar{b} \to H$ [Neubert et al. 19-22]
  - ▶ Thrust distribution [Stewart et al. 19; Beneke et al. 22]
  - ▶ Leptonic $B$ decays [Feldmann, Gubernari, Huber, Seitz 22; Cornella et al. 22-26]
  - ▶ Resolved contribution in $B \to X_s\gamma$ [Hurth, Szafron 22]
  - ▶ $e\mu$ backward scattering [Bell, Böer, Feldmann 22]
  - ▶ Virtual Compton scattering [Schoenleber, Szafron 25]

- ▶ At next-to-leading power endpoint-divergent convolution integrals arise generically

- ▶ Novel refactorization-based ideas [Böer 18; Neubert, Liu 19] do not seem applicable to exclusive hadronic $B$ decays (e.g. $B \to \pi\ell\nu$, $B \to \pi\pi$)

# Exclusive $B$ decays at large recoil

$$\langle \pi(p) | \bar{q}\gamma^\mu b | \bar{B}(p_B) \rangle = F_+(q^2)(p_B^\mu + p^\mu) + F_-(q^2)q^\mu$$

$$F_i(q^2) = C_i \xi_{B\to\pi}(q^2) + \phi_B \otimes T_i^I(q^2) \otimes \phi_\pi$$

[Beneke, Feldmann 00]
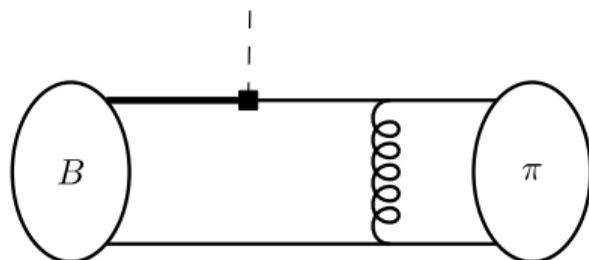


▶ Naive factorization theorem is ill-defined because of endpoint-divergences → soft-overlap

[Bauer, Pirjol, Stewart 02; Beneke, Feldmann 03; Neubert, Lange 03]

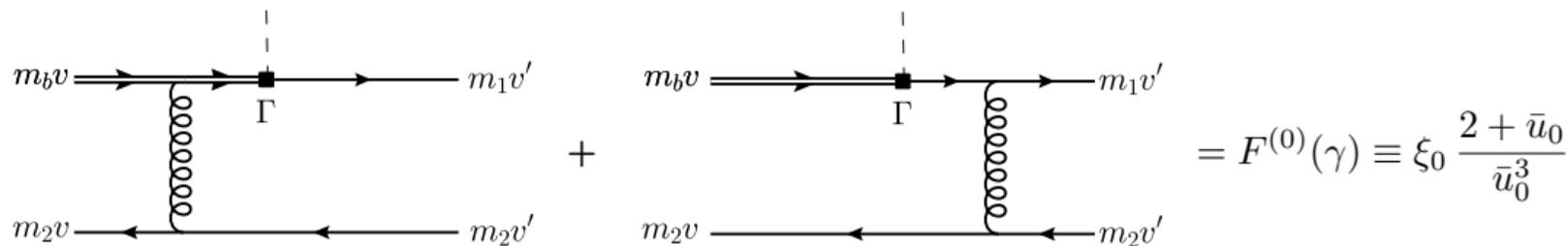$$\xi_{B\to\pi}(q^2) = \sum_i \int_0^\infty \mathrm{d}\omega \int_0^1 \mathrm{d}u \ \phi_B^i(\omega) T_{II}^i(u,\omega;q^2)\phi_\pi^i(u)$$

▶ This is a long-standing problem since the early days of SCET

# Non-relativistic heavy-to-light form factors

▶ Process considered:

    ▶ $B_c \to \eta_c$ at large recoil $\gamma \equiv v \cdot v' = \mathcal{O}(m_b/m_c)$

    ▶ Non-relativistic approximation with $m_b \gg m_c \gg \Lambda_{\text{QCD}}$

    ▶ Perturbative toy example for $B \to \pi$ form factors

▶ Define soft-overlap form factor

$$F(\gamma) \equiv \frac{1}{2E_\eta} \langle \eta_c(p_\eta)|(\bar{q}_1 \, \Gamma \, b)(0)|B_c(p_B)\rangle \qquad \text{with} \qquad \Gamma = \frac{\slashed{p}\slashed{\eta}}{4}$$



$$= F^{(0)}(\gamma) \equiv \xi_0 \, \frac{2 + \bar{u}_0}{\bar{u}_0^3}$$
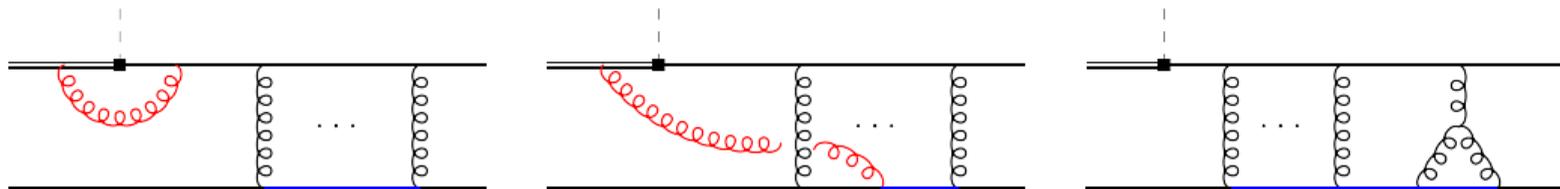
$$\xi_0 = \frac{\alpha_s C_F}{4\pi} \frac{\pi^2 f_\eta f_B m_B}{N_c E_\eta^2 m_\eta}, \quad \bar{u}_0 = \frac{m_2}{m_\eta}$$

# Diagrammatic resummation of double logarithms

Combining effects from soft gluons and soft quarks gives
iterated integral equations featuring Sudakov factors

$$\left(S(x,y) \equiv \frac{\alpha_s C_F}{2\pi} \ln x \ln y\right)$$

[Bell, Böer, Feldmann, DH, Shtabovenko 24]



$$F(\gamma) = \xi_0 \exp\left\{-\frac{\alpha_s C_F}{4\pi} L^2\right\} \left(2\frac{1+\bar{u}_0}{\bar{u}_0^3} f_2(m_2,m_2) + \frac{C_A}{2C_F \bar{u}_0^3}\left(1 - f_1(m_2,m_2)\right) - \frac{1}{\bar{u}_0^2}\right)$$

$$f_1(q_+,q_-) = 1 + \frac{\alpha_s}{4\pi} \int\limits_{q_-}^{p_{2-}} \frac{\mathrm{d}k_-}{k_-} \int\limits_{m_2^2/k_-}^{q_+} \frac{\mathrm{d}k_+}{k_+} \exp\left\{-S\left(\frac{q_+}{k_+},\frac{p_{\eta-}}{k_-}\right)\right\}\left[2C_F f_1(k_+,k_-)\right]$$

$$f_2(q_+,q_-) = 1 + \frac{\alpha_s}{4\pi} \int\limits_{q_-}^{p_{2-}} \frac{\mathrm{d}k_-}{k_-} \int\limits_{m_2^2/k_-}^{q_+} \frac{\mathrm{d}k_+}{k_+} \exp\left\{-S\left(\frac{q_+}{k_+},\frac{p_{\eta-}}{k_-}\right)\right\}\left[2C_F f_2(k_+,k_-) + \left(C_F - \frac{C_A}{2}\right)f_1(k_+,k_-) + \frac{C_A}{2}\right]$$

# SCET analysis: QCD → SCET-1 matching

- Need to set up a factorization theorem that covers the scales $m_b \gg \sqrt{m_b m_\eta} \gg m_\eta$
- First step: match QCD → SCET-1 to integrate out the hard scale $m_b$
- Form factor $F$ chosen specifically such that only $A$-type current contributes

$$J_A = \bar{\chi}_{hc} h_v$$

- It follows

$$F(q^2) = H(m_b, E_\eta)\, \xi(q^2)$$

with

$$\xi(q^2) = \frac{1}{2E_\eta} \langle \eta_c(p')|\, J_A(0)\, |B_c(v)\rangle \quad \text{and} \quad H(m_b, E_\eta) \simeq 1 - \frac{\alpha_s C_F}{4\pi} \frac{1}{\varepsilon^2} \left(\frac{\mu^2}{4E_\eta^2}\right)^\varepsilon + \mathcal{O}(\alpha_s^2, 1/\epsilon)$$

# SCET analysis: SCET-1 $\rightarrow$ SCET-2 matching

▶ Second step: match SCET-1 $\rightarrow$ SCET-2 to integrate out intermediate scale $\sqrt{m_b m_\eta}$

▶ $J_A$ matches onto 5 operators in SCET-2: [Lange, Neubert 04; Böer 18]

$$\mathcal{O}_1 = g_s^2 \left[ \bar{\chi}_c(0) \frac{\not{\bar{n}}}{2} \gamma_5 \chi_c(s\bar{n}) \right] \left[ \bar{\mathcal{Q}}_s(\tau n) \frac{\not{\bar{n}}\not{n}}{4} \gamma_5 \mathcal{H}_v(0) \right]$$

$$\mathcal{O}_2 = g_s^2 \left[ \bar{\chi}_c(0) \frac{\not{\bar{n}}}{2} \gamma_5 i\not{\partial}_\perp \chi_c(s\bar{n}) \right] \left[ \bar{\mathcal{Q}}_s(\tau n) \frac{\not{n}}{2} \gamma_5 \mathcal{H}_v(0) \right]$$

$$\mathcal{O}_3 = g_s^2 \left[ \bar{\chi}_c(0) \frac{\not{\bar{n}}}{2} \gamma_5 \not{\mathcal{A}}_{c,\perp}(r\bar{n}) \chi_c(s\bar{n}) \right] \left[ \bar{\mathcal{Q}}_s(\tau n) \frac{\not{n}}{2} \gamma_5 \mathcal{H}_v(0) \right]$$

$$\mathcal{O}_4 = g_s^2 \left[ \bar{\chi}_c(0) \frac{\not{\bar{n}}}{2} \gamma_5 \chi_c(s\bar{n}) \right] \left[ \bar{\mathcal{Q}}_s(\tau n) \not{\mathcal{A}}_{s,\perp}(\sigma n) \frac{\not{n}}{2} \gamma_5 \mathcal{H}_v(0) \right]$$

$$\mathcal{O}_m = g_s^2 \left[ \bar{\chi}_c(0) \frac{\not{\bar{n}}}{2} \gamma_5 \chi_c(s\bar{n}) \right] \left[ \bar{\mathcal{Q}}_s(tn) \frac{\not{n}}{2} \gamma_5 \mathcal{H}_v(0) \right]$$

▶ (Tree-level) matching coefficients read, e.g.:

$$D_1^{(0)}(u,\omega) = -\frac{C_F}{N_C} \frac{1+\bar{u}}{4E^2 \bar{u}^2 \omega}, \qquad D_m^{(0)}(u,\omega) = \frac{C_F}{N_C} \frac{1}{4E^2 \bar{u} \, \omega^2} \left( m_1 \frac{\bar{u}}{u} + m_2 \frac{u}{\bar{u}} \right)$$

# SCET analysis: Hadronic matrix elements & factorization formula

▶ Hadronic soft and collinear matrix elements introduce LCDAs of the $B$ and $\eta_c$ meson, e.g.:

$$\langle 0| \, \bar{\mathcal{Q}}_s(\tau n)\slashed{n}\gamma_5 \mathcal{H}_v(0) \, |B_c(v)\rangle = i f_B m_B \int_0^\infty d\omega \, e^{-i\omega\tau} \, \phi_B^+(\omega)$$

$$\langle \eta_c(p')| \, \bar{\chi}_c(0)\slashed{\bar{n}}\gamma_5\chi_c(s\bar{n}) \, |0\rangle = -i \, 2E_\eta \, f_P \int_0^1 du \, e^{2iE_\eta s\bar{u}}\phi(u)$$

▶ Restricting to tree-level hard-collinear matching gives the factorization formula [Böer 18]

$$\begin{aligned}
\xi^{\mathrm{LO}}(q^2) = \xi_0 m_\eta \Bigg\{ &\int_0^\infty d\omega \int_0^1 du \left[ \frac{\phi_B^-(\omega) + \phi_B^+(\omega)}{\omega} \frac{1+\bar{u}}{\bar{u}^2} \phi(u) - 2\frac{\phi_B^+(\omega)}{\omega} \frac{\phi(u)}{\bar{u}} \right. \\
&+ \left. \frac{\phi_B^+(\omega)}{\omega^2} \left( -m_1\frac{\phi(u)}{\bar{u}} - 2m_2\frac{\phi(u)}{\bar{u}^2} + 3\mu_P\frac{\phi_p(u)}{\bar{u}} + \frac{\tilde{\mu}_P}{6}\frac{\phi'_\sigma(u)}{\bar{u}} \right) \right] \\
&- \frac{C_A - 2C_F}{C_F} \frac{f_{3P}}{f_P} \int_0^\infty d\omega \frac{\phi_B^+(\omega)}{\omega^2} \int \mathcal{D}\alpha \frac{\phi_3(\{\alpha_i\})}{\alpha_2\alpha_3(\alpha_2 + \alpha_3)} \\
&+ \frac{C_A - 2C_F}{C_F} \int_0^\infty d\omega \int_0^\infty d\xi \frac{\phi_{3B}(\omega,\xi)}{\omega\xi(\omega + \xi)} \int_0^1 du \frac{\phi(u)}{\bar{u}^2} \Bigg\}
\end{aligned}$$

# SCET analysis: results at 1-loop

▶ Factorization formula contains endpoint-divergent moments, e.g.:

$$\phi_B^+(\omega;\mu) \sim \omega \qquad \Longrightarrow \qquad \int_0^\infty d\omega \, \frac{m_2^2}{\omega^2} \, \phi_{B,\text{reg}}^{+,(1)}(\omega) \qquad \text{is endpoint-divergent}$$

▶ Introduce rapidity regulator $d^D k \to d^D k \, (\nu/k_-)^\alpha$                                           [Becher, Bell 11]

$$\int_0^\infty d\omega \, \frac{m_2^2}{\omega^2} \, \phi_{B,\text{reg}}^{+,(1)}(\omega) \simeq 2C_F \left( \frac{1}{\alpha\varepsilon} - \frac{1}{\varepsilon^2} \right) \left( \frac{\mu^2}{m_2^2} \right)^\varepsilon \left( \frac{\nu}{m_2} \right)^\alpha - \frac{C_F}{\varepsilon^2} \left( \frac{\mu^2}{m_2^2} \right)^\varepsilon$$

▶ Combining all double poles arising from hard(-collinear) matching coefficients, hadronic matrix elements, and endpoint divergences, one finds that the poles cancel, leaving behind the double logarithms                                                                   [Bell 06, Böer 18]

$$F^{(1)}(\gamma) \simeq \xi_0 L^2 \left( C_F \frac{1+2\bar{u}_0}{\bar{u}_0^3} - \frac{C_A}{2\bar{u}_0^3} \right), \quad \text{with} \quad L \equiv \ln(2\gamma)$$

## Independent check up to 3-loops

▶ Computing matching coefficients and hadronic matrix elements explicitly becomes impractical at higher orders

▶ Turn argument around and use pole cancellation. Ansatz:

$$F_h^{(1)}(\gamma) \simeq \frac{h_{20}}{\epsilon^2} \left(\frac{\mu^2}{4E_\eta^2}\right)^\epsilon F^{(0)}(\gamma), \qquad\qquad F_j^{(1)}(\gamma) \simeq \frac{j_{20}}{\epsilon^2} \left(\frac{\mu^2}{2m_\eta E_\eta}\right)^\epsilon,$$

$$F_c^{(1)}(\gamma) \simeq \left\{\frac{c_{20}}{\epsilon^2} + \frac{c_{11}}{\alpha\epsilon}\right\} \left(\frac{\mu^2}{m_\eta^2}\right)^\epsilon \left(\frac{\nu}{2E_\eta}\right)^\alpha, \qquad F_s^{(1)}(\gamma) \simeq \left\{\frac{s_{20}}{\epsilon^2} + \frac{s_{11}}{\alpha\epsilon}\right\} \left(\frac{\mu^2}{m_\eta^2}\right)^\epsilon \left(\frac{\nu}{m_\eta}\right)^\alpha.$$

▶ Imposing pole cancellation gives

$$F^{(1)}(\gamma) \simeq \left(2h_{20}F^{(0)}(\gamma) + \frac{j_{20}}{2}\right)L^2$$

▶ At 2- and 3-loop the only unknown coefficients can be chosen to be those of the purely hard-collinear region: $j_{40}$ and $j_{60}$, respectively

# Computation of missing hard-collinear coefficients

▶ Essentially massless five-point function with simplified kinematics and eikonal propagators

▶ Generate diagrams with QGRAF [Nogueira 93]

▶ Insert Feynman rules and simplify Dirac structure in Form [Vermaseren, 2000; Kuipers et al., 2013]

▶ Identify topologies with the help of FeynCalc 10 [Shtabovenko et al. 23]

▶ IBP reduction to obtain MIs with Fire [Smirnov et al. 20] and Kira [Lange et al. 25]

▶ Evaluate MIs numerically with pySecDec [Borowka et al. 18]
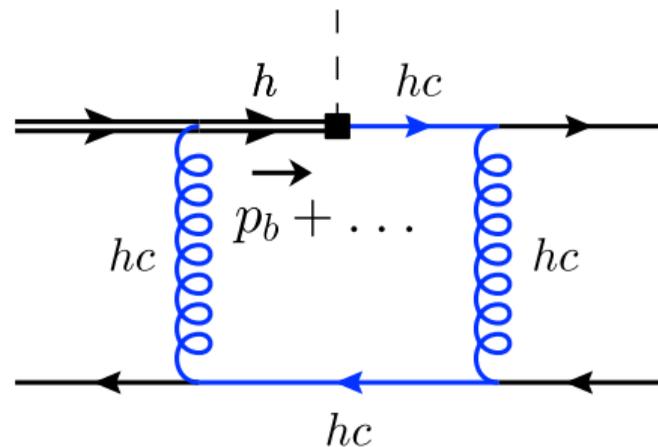


| # of loops | # contributing master integrals |
|:---:|:---:|
| 1 | 3 |
| 2 | 46 |
| 3 | 1671 |

# Results

▶ At 2-loop the numerical precision is sufficient to reconstruct the result analytically

$$j_{40} = \frac{\xi_0 C_F}{\bar{u}_0^3} \left( C_F \left( 17 + 15\bar{u}_0 \right) + C_A \left( -\frac{5}{2} - \frac{\bar{u}_0}{2} \right) \right)$$

▶ At 3-loop there is an additional complication in the form of spurious poles up to $1/\varepsilon^{11}$

▶ The result for $j_{60}$ that would be compatible with our integral equations is

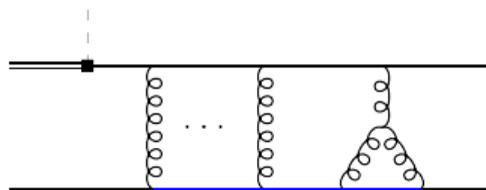$$j_{60} = \frac{\xi_0 C_F^2}{\bar{u}_0^3} \left( C_F \frac{148 + 136\bar{u}_0}{9} - C_A \frac{26 + 8\bar{u}_0}{9} \right)$$
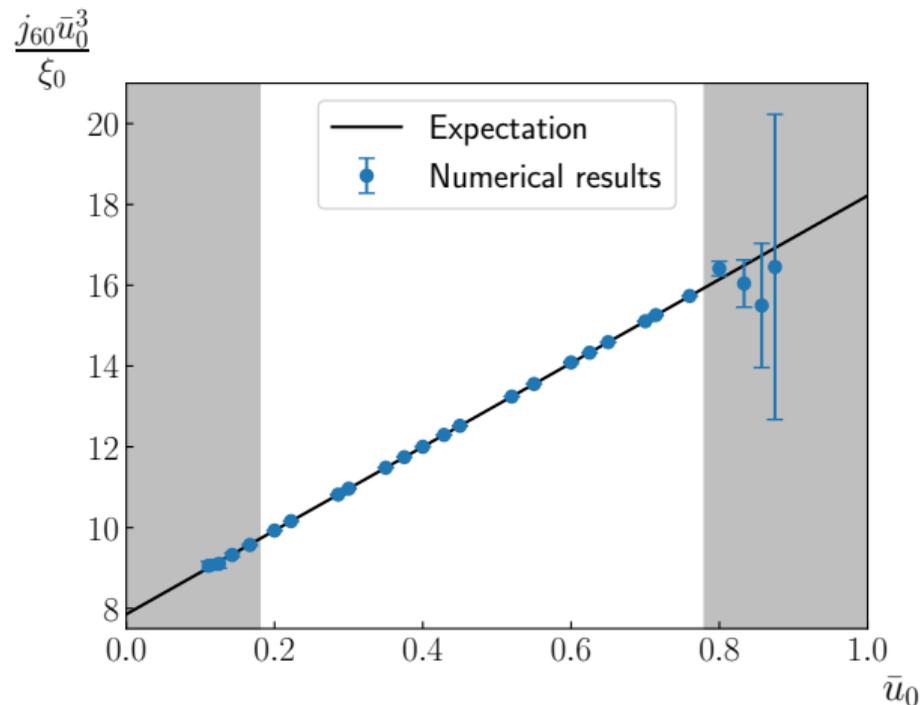


$$j_{60}\bar{u}_0^3/\xi_0 = 7.857 \quad + 10.359\,\bar{u}_0$$

$$\left( j_{60}\bar{u}_0^3/\xi_0 \right)_{\text{fit}} = 7.856(5) + 10.366(13)\,\bar{u}_0 \quad \checkmark$$

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# Resummation via cut-off moments and RG equations



$$\longrightarrow f_1(q_+, q_-) = 1 + \frac{\alpha_s}{4\pi} \int\limits_{q_-}^{p_{2-}} \frac{\mathrm{d}k_-}{k_-} \int\limits_{m_2^2/k_-}^{q_+} \frac{\mathrm{d}k_+}{k_+} \, e^{-S\left(\frac{q_+}{k_+}, \frac{p_{\eta-}}{k_-}\right)} \Big[ 2C_F \, f_1(k_+, k_-) \Big]$$

▶ Iterative integral equations can be rewritten in terms of differential equations, e.g. for $f_1$

$$(\partial_\rho \partial_\eta + \eta \partial_\eta - 1) \, g_1(\rho, \eta) = 0, \qquad \text{with} \qquad \rho \sim \ln\left(\frac{q_+ p_-}{m_2^2}\right) \quad \text{and} \quad \eta \sim \ln\left(\frac{p_-}{q_-}\right)$$

▶ Endpoint divergences are a consequence of overlapping soft and collinear regions

$\implies$ An appropriate cut-off in the soft or collinear region captures the relevant physics

▶ $\phi_B^+$ (and $\phi_B^-$) fulfill RG equations  [Lange, Neubert 03]

$$\frac{d}{d\ln\mu} \phi_B^+(\omega; \mu) = -\int_0^\infty d\omega' \, \gamma_+(\omega, \omega'; \mu) \phi_B^+(\omega'; \mu)$$

# Resummation via cut-off moments and RG equations

▶ Focusing on terms in $\gamma_+$ that give double logs at the endpoint gives

$$\frac{d}{d \ln \mu} \phi_B^+(\omega; \mu) \simeq -\Gamma_{\rm cusp}(\alpha_s) \left( \ln \frac{\mu}{\omega} \phi_B^+(\omega; \mu) - \omega \int_\omega^\infty \frac{d\omega'}{\omega'^2} \phi_B^+(\omega'; \mu) \right)$$

▶ Cut-off moment fulfills the same DE as $f_1$

$$g^+(\kappa; \mu) \equiv \int_\kappa^\infty \frac{d\omega}{\omega^2} \tilde{\phi}_B^+(\omega; \mu)$$

$$\implies \qquad \left( \frac{d^2}{d \ln \kappa \, d \ln \mu} - \frac{\alpha_s C_F}{\pi} \ln \frac{\kappa}{\mu_0} \frac{d}{d \ln \kappa} + \frac{\alpha_s C_F}{\pi} \right) g^+(\kappa; \mu) = 0$$

▶ Similar derivation leads to DE for cut-off moment of $\phi_B^-$ that corresponds to DE for $f_2$

# Summary and outlook

- Exclusive B decays play important role in flavor physics phenomenology
- QCD dynamics very involved
    - Studied $B_c \to \eta_c$ form factor at large recoil in NR approximation
    - Complicated interplay of soft gluon and soft quark dynamics
    - Double logs are governed by coupled recursive integral equations
- We investigated how this double-log structure is reproduced in SCET
    - Starting point is an (endpoint-divergent) bare factorization formula
    - With regulators in place this verifies the integral equations up to three-loop order
- Recursive integral equations can be interpreted in terms of cut-off moments of LCDAs

## Backup: Resummation via cut-off moments and RG equations

Lange-Neubert kernel:

$$\gamma_+(\omega, \omega'; \mu) = \left[ \Gamma_{\text{cusp}}(\alpha_s) \log \frac{\mu}{\omega} + \gamma(\alpha_s) \right] \delta(\omega - \omega') - \Gamma_{\text{cusp}}(\alpha) \, \omega \Gamma(\omega, \omega')$$

$$\Gamma(\omega, \omega') = \left[ \frac{\theta(\omega - \omega')}{\omega(\omega - \omega')} + \frac{\theta(\omega' - \omega)}{\omega'(\omega' - \omega)} \right]_+$$

Separate the contribution of the universal cusp

$$\phi_B^+(\omega; \mu) \equiv \exp\left\{ -\frac{\alpha_s C_F}{2\pi} \ln^2 \frac{\mu}{\mu_0} \right\} \tilde{\phi}_B^+(\omega; \mu)$$

$$\implies \qquad \frac{d}{d \ln \mu} \tilde{\phi}_B^+(\omega; \mu) \simeq \frac{\alpha_s C_F}{\pi} \left( \log \frac{\omega}{\mu_0} \, \tilde{\phi}_B^+(\omega; \mu) + \omega \int_\omega^\infty \frac{d\omega'}{\omega'^2} \tilde{\phi}_B^+(\omega'; \mu) \right)$$