NNLO Soft Function for 0-jettiness in $t\bar{t}$ Production

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March 2nd, 2026

In collaboration with
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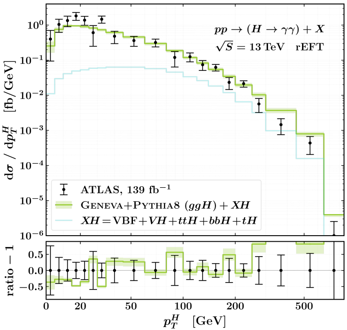
SCET 2026
March 2nd - 5th, KIAS

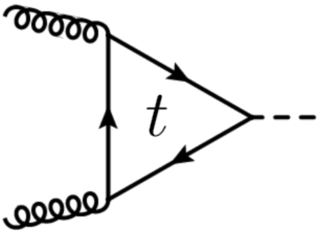
Outline

- Motivation
- Framework
- NLO Soft function
- NNLO Soft function
- Renormalization
- Outlook & Summary

- Monte Carlo event generator
- Combines fixed order, resummation and Parton Shower (PS)



[S. Alioli; '21]



- Multitude of colour neutral processes already implemented at NNLO+PS accuracy
  - $\Rightarrow$ Our goal consists in extending *GENEVA* to processes with top quarks in final state

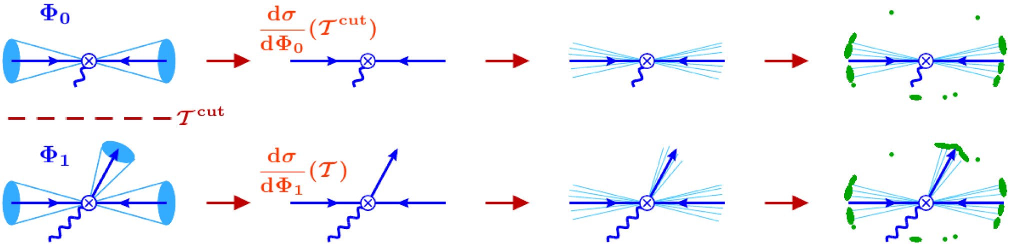
N-jettiness observable

[Stewart, Tackmann, Waalewijn; ’10]

$$\tau_N = \sum_k \min\left\{ p_k^+,\, p_k^-,\, n_1 \cdot p_k,\, \ldots,\, n_N \cdot p_k \right\}$$

small values of τ_N indicate N-jet-like event

- N-jettiness is used in *GENEVA* as a jet resolution parameter



- First automated calculation of dijet soft functions with SoftServe

  [Bell, Rahn, Talbert; '18]

- Automated calculation of N-jettiness soft function [Bell et al; '23]

- Automation of jet and beam functions at NNLO [Bell et al; '24][Brune; PhD thesis]

- So far only for massless final states

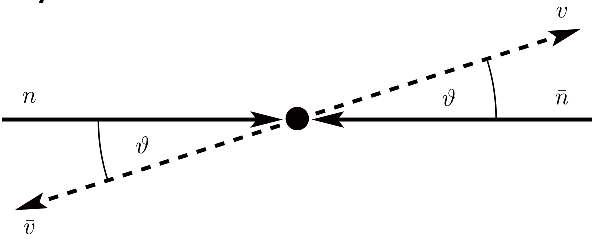
  $\Rightarrow$ Extension to massive final state

$$\frac{d\sigma}{d\phi_0 d\tau_0} = \sum_{\text{m, n}\in\{q\bar{q},\bar{q}q,gg\}} \int dt_a dt_b \, \boldsymbol{B}^{\text{m}}(t_a,\, z_a,\, \mu)\boldsymbol{B}^{\text{n}}(t_b,\, z_b,\, \mu)$$

$$\times \boldsymbol{H}_{JI}^{\text{mn}\to t\bar{t}}(\phi_0,\, \mu)\boldsymbol{S}_{IJ}^{\text{mn}\to t\bar{t}}\left(M\tau_0 - \frac{t_a + t_b}{M},\, \phi_0,\, \mu\right)$$

- The 0-jettiness cross section "factorizes" for small τ_0 [Alioli, Broggio, Lim; '21]
  - ⇒ hard function $\boldsymbol{H}_{JI}^{\text{mn}}$ contains purely virtual extensions of Born amplitude
  - ⇒ beam functions $\boldsymbol{B}^{\text{i}}$ describe collinear emissions along beam direction $n, \bar{n}$
  - ⇒ **soft function** is still missing at NNLO

$t\bar{t}$ production in strictly back-to-back case



$$n^\mu = (1,\, 0,\, 0,\, 1) \qquad \bar{n}^\mu = (1,\, 0,\, 0,\, -1)$$

$$\bar{v}^\mu(\vartheta,\, \beta) = \frac{p_{\bar{t}}^\mu}{m_t} \qquad v^\mu(\vartheta,\, \beta) = \frac{p_t^\mu}{m_t}$$

$\beta \equiv$ Velocity of t quarks

$\vartheta \equiv$ Scattering angle of t quarks

$$S(\kappa,\,\mu) = \sum_{X_s} \mathcal{M}(\kappa;\,\{k_i\}) \left| \left\langle X_s \right| \underbrace{\mathcal{S}_n^\dagger(0)\mathcal{S}_{\bar{n}}(0)\mathcal{S}_v^\dagger(0)\mathcal{S}_{\bar{v}}(0)}_{\text{Soft Wilson lines}} \left| 0 \right\rangle \right|^2$$

$$= 1 + \left(\frac{Z_\alpha \alpha_s}{4\pi}\right)\left(\frac{\mu^2}{\kappa^2}\right)^{\varepsilon} S^{(1)}(\varepsilon) + \left(\frac{Z_\alpha \alpha_s}{4\pi}\right)^2 \left(\frac{\mu^2}{\kappa^2}\right)^{2\varepsilon} S^{(2)}(\varepsilon) + \ldots$$

Partonic dijet soft functions

$$S(\kappa,\, \mu) = \sum_{X_s} \mathcal{M}(\kappa;\, \{k_i\}) \left| \left\langle X_s \right| \underbrace{\mathcal{S}_n^\dagger(0)\mathcal{S}_{\bar{n}}(0)\mathcal{S}_v^\dagger(0)\mathcal{S}_{\bar{v}}(0)}_{\text{Soft Wilson lines}} \left| 0 \right\rangle \right|^2$$

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- For massive final-state partons so far only known at NLO [Alioli, Broggio, Lim; '21], [Münker; M thesis]

$$S(\kappa,\,\mu) = \sum_{X_s} \mathcal{M}(\kappa;\,\{k_i\}) \left| \left\langle X_s \right| \underbrace{\mathcal{S}_n^\dagger(0)\mathcal{S}_{\bar{n}}(0)\mathcal{S}_v^\dagger(0)\mathcal{S}_{\bar{v}}(0)}_{\text{Soft Wilson lines}} \left| 0 \right\rangle \right|^2$$

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$$S^{(2)}(\varepsilon) = S^{(2,\,\mathrm{RV})}(\varepsilon) + S^{(2,\,q\bar{q})}(\varepsilon) + S^{(2,\,gg)}(\varepsilon)$$

- For massive final-state partons so far only known at NLO [Alioli, Broggio, Lim; '21], [Münker; M thesis]
- Automated calculation of NNLO soft function with **massless** partons [Bell, Rahn, Talbert; '19]
- NNLO soft functions contain both real-virtual corrections and double-real emissions

- Measurement function is exponential of soft emission momenta k_i

$$\mathcal{M}(\kappa;\,\{k_i\}) = \exp\left[-\frac{\tau(\{k_i\})}{\kappa}\right]$$

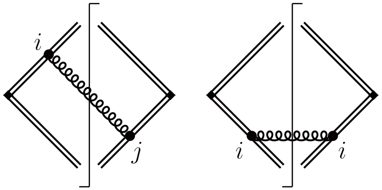
- For one soft emission k, the measurement function can be described by the ansatz

$$\mathcal{M}_1 = \exp\left[-\frac{k_T}{\kappa}\left(y_k^{n/2} f_A(y_k,\,t_k)\theta(1-y_k) + y_k^{-n/2} f_B(y_k^{-1},\,t_k)\theta(y_k-1)\right)\right]$$

- We use the phase space parametrization

$$y_k = \frac{k_+}{k_-} \qquad k_T = \sqrt{k_+ k_-} \qquad t_k = \frac{1-\cos\theta_k}{2}$$

NLO soft function with massive partons



$$S_{ij}^{(1)} = -\frac{2e^{-\gamma_E \varepsilon}\,\Gamma(-2\varepsilon)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\varepsilon\right)} \int_0^1 \frac{\mathrm{d}y_k}{y_k^{1-n\varepsilon}} \int_0^1 \frac{\mathrm{d}t_k}{[4t_k(1-t_k)]^{\frac{1}{2}+\varepsilon}}$$

$$\times \left\{ [f_A(y_k, t_k)]^{2\varepsilon}\, \mathcal{J}_{ij}^{(1)}(y_k, t_k) + [f_B(y_k, t_k)]^{2\varepsilon}\, \mathcal{J}_{ij}^{(1)}\left(\frac{1}{y_k}, t_k\right) \right\}$$

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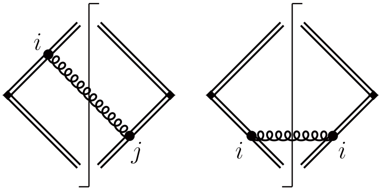

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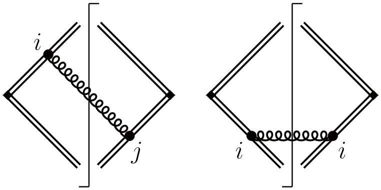
soft divergence

NLO soft function with massive partons



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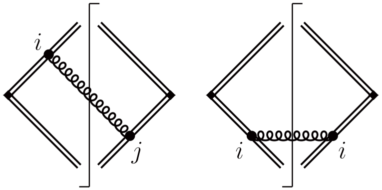
soft divergence

collinear divergence

NLO soft function with massive partons



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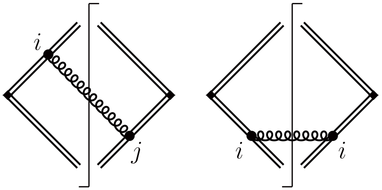
soft divergence

collinear divergence

observable dependent

NLO soft function with massive partons



$$\mathcal{J}_{ij}^{(1)} = \frac{p_i \cdot p_j}{(p_i \cdot k)(p_j \cdot k)}$$

$$S_{ij}^{(1)} = -\frac{2e^{-\gamma_E \varepsilon}\,\Gamma(-2\varepsilon)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\varepsilon\right)} \int_0^1 \frac{\mathrm{d}y_k}{y_k^{1-n\varepsilon}} \int_0^1 \frac{\mathrm{d}t_k}{[4t_k(1-t_k)]^{\frac{1}{2}+\varepsilon}}$$

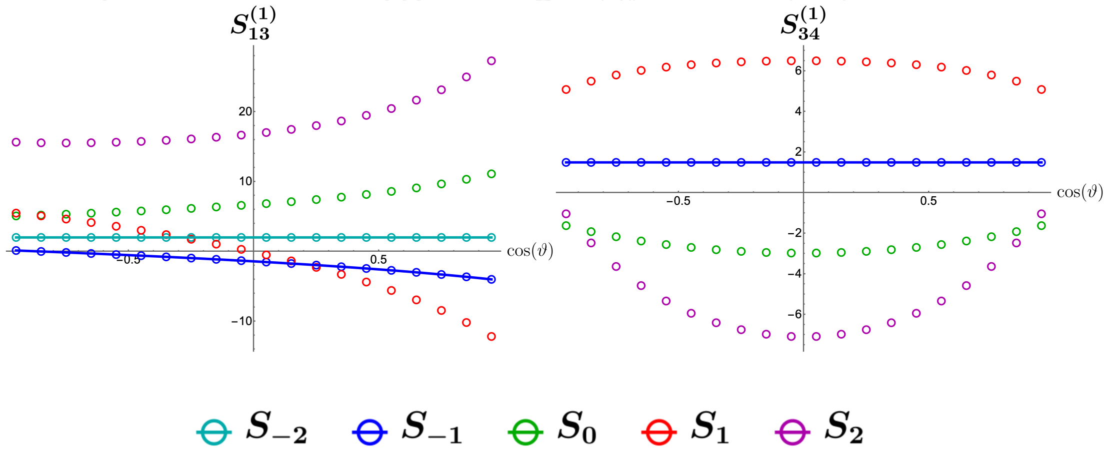
$$\times \left\{ [f_A(y_k, t_k)]^{2\varepsilon}\, \mathcal{J}_{ij}^{(1)}(y_k, t_k) + [f_B(y_k, t_k)]^{2\varepsilon}\, \mathcal{J}_{ij}^{(1)}\left(\frac{1}{y_k}, t_k\right) \right\}$$

soft divergence

collinear divergence

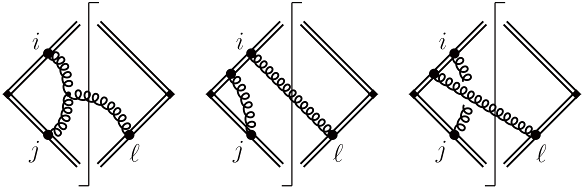
observable dependent

matrix element



$$S(\cos\vartheta,\ \beta,\ \varepsilon) \equiv \frac{S_{-2}}{\varepsilon^2} + \frac{S_{-1}}{\varepsilon} + S_0 + \varepsilon \times S_1 + \varepsilon^2 \times S_2 + \mathcal{O}(\varepsilon^3)$$

Real-virtual corrections

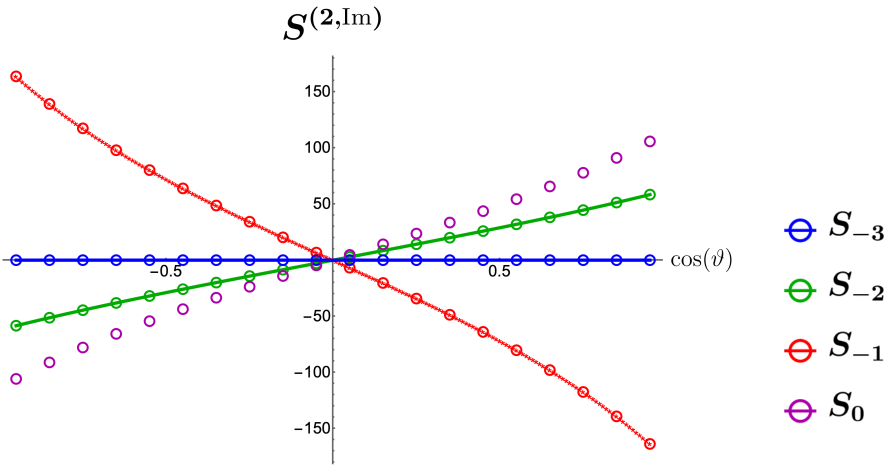


$$S^{(2,\,\mathrm{RV})}(\varepsilon) = C_A \sum_{i\neq j} \boldsymbol{T}_i \cdot \boldsymbol{T}_j S_{ij}^{(2,\,\mathrm{Re})}(\varepsilon) + \sum_{i\neq j\neq \ell} f^{ABC} \boldsymbol{T}_i^A \boldsymbol{T}_j^B \boldsymbol{T}_\ell^C S_{ij\ell}^{(2,\,\mathrm{Im})}(\varepsilon)$$

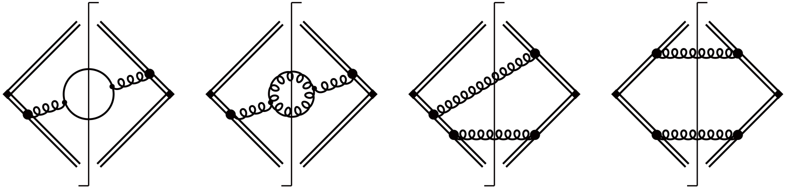
- Real-virtual matrix elements with massive partons are known
  [Bierenbaum, Czakon, Mitov; '12], [Czakon, Mitov; '18]

  $\Rightarrow$ suitable for numerical calculation

Tripole results

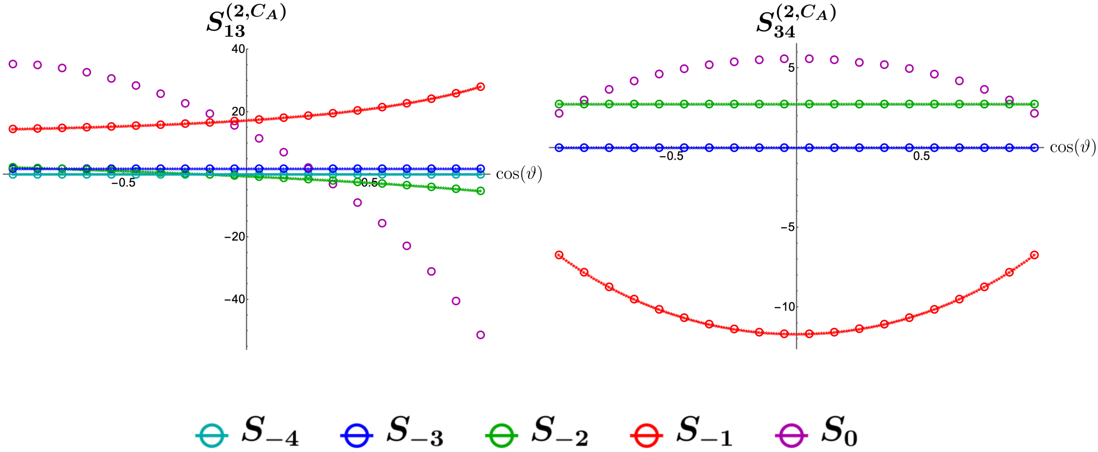


- Pole prediction only possible for sum over all tripoles



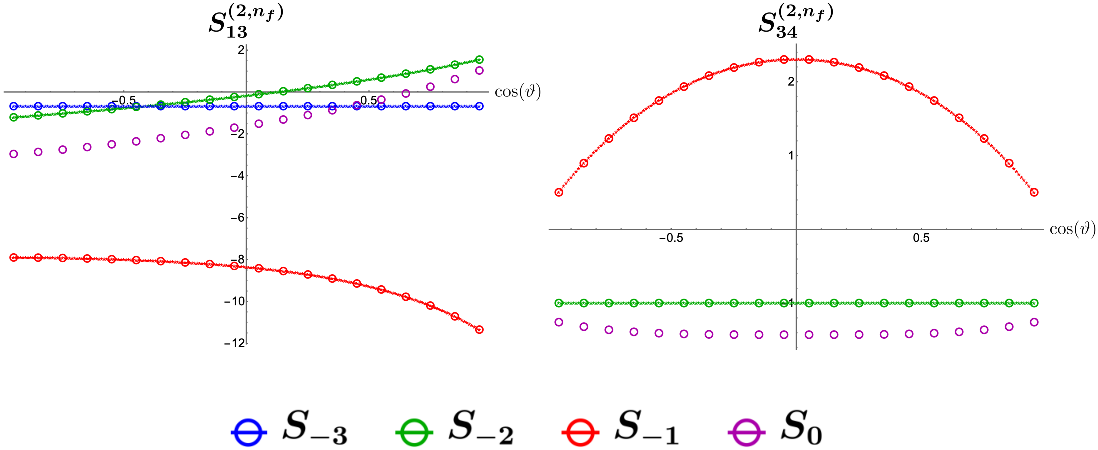
- Matrix elements already known [Angeles-Martinez, Czakon, Sapeta; ’18]
- Parametrizations of massless case can be carried over [Bell, Rahn, Talbert; ’19]
  - ⇒ factorization of phase space divergences more complicated
  - ⇒ beam-beam-parametrization simplifies calculation in transverse phase space

Full C_A results



- Both real-virtual and double-real contributions have to be added for C_A pole prediction

n_f results for completion



- $\boldsymbol{S_R}$ is matrix in colour space and renormalizes as

$$\boldsymbol{S_R} = \boldsymbol{Z}^\dagger \boldsymbol{S_b} \boldsymbol{Z} = \left(e^{\ln \boldsymbol{Z}}\right)^\dagger e^{\ln \boldsymbol{S_b}} e^{\ln \boldsymbol{Z}}$$

$$\ln \boldsymbol{S_R} = \left(\frac{Z_\alpha \alpha_s}{4\pi}\right)\left(\boldsymbol{S_b}^{(1)} + \boldsymbol{Z}^{(1)} + \boldsymbol{Z}^{\dagger\,(1)}\right) + \left(\frac{Z_\alpha \alpha_s}{4\pi}\right)^2 \Big[\ln \boldsymbol{S_b}^{(2)} + \ln \boldsymbol{Z}^{(2)} + \ln \boldsymbol{Z}^{\dagger\,(2)} + \frac{1}{2}\left(\left[\boldsymbol{S_R}^{(1)}, \left(\boldsymbol{Z}^{(1)} - \boldsymbol{Z}^{\dagger\,(1)}\right)\right] + \left[\boldsymbol{Z}^{\dagger\,(1)}, \boldsymbol{Z}^{(1)}\right]\right)\Big] + \mathcal{O}(\alpha_s^3)$$

- Massive NNLO anomalous dimension already known [Becher, Neubert; '09] [Ferroglia et al; '09]

$$\boldsymbol{\Gamma^S}(\kappa, \mu) = 2\boldsymbol{Z}^{(-1)} \frac{\mathrm{d}\boldsymbol{Z}}{\mathrm{d}L} = \sum_{i,j} \frac{\boldsymbol{\mathcal{T}}_i \cdot \boldsymbol{\mathcal{T}}_j}{2}\left[-\Gamma L - \Gamma \ln\left(\frac{M_{t\bar{t}}^2}{-s_{ij} - i0}\right) + \gamma^{S,\,ij}\right]$$
$$+ \sum_{I,j} \frac{\boldsymbol{\mathcal{T}}_I \cdot \boldsymbol{\mathcal{T}}_j}{2}\left[-\frac{\Gamma}{2} L - \Gamma \ln\left(\frac{m_t M_{t\bar{t}}}{-s_{Ij}}\right) + \gamma^{S,\,Ij}\right] + \sum_{IJ} \frac{\boldsymbol{\mathcal{T}}_I \cdot \boldsymbol{\mathcal{T}}_J}{2}\left[\gamma(\beta_{IJ}) + \gamma^{S,\,IJ}\right]$$
$$+ \sum_{IJ}\sum_{k} i f^{ABC} \boldsymbol{\mathcal{T}}_I^A \boldsymbol{\mathcal{T}}_J^B \boldsymbol{\mathcal{T}}_k^C f_2\left(\beta_{IJ}, \ln\frac{-\sigma_{Jk} v_J \cdot p_k}{-\sigma_{Ik} v_I \cdot p_k}\right) + \mathcal{O}(\alpha_s^2)$$

- Tripole contributions originate from both the commutators of dipoles and genuine tripole contribution to Z

$$\frac{1}{2}\left[\boldsymbol{Z}^{(1)},\, \boldsymbol{Z}^{\dagger\,(1)}\right] = \left(\frac{\alpha_s}{4\pi}\right)^2 f^{ABC}\boldsymbol{T}_1^A\boldsymbol{T}_2^B\boldsymbol{T}_3^C\left[\frac{\Gamma_0^2}{2\varepsilon^2}\pi\ln\left(\frac{-s_{31}}{-s_{32}}\right) + \frac{\pi\Gamma_0}{\varepsilon^2}\frac{1+\beta^2}{\beta}\ln\left(\frac{-s_{31}}{-s_{32}}\right)\right]$$

$$\frac{1}{2}\left[\boldsymbol{S_R}^{(1)},\, (\boldsymbol{Z}^{(1)} - \boldsymbol{Z}^{\dagger\,(1)})\right] = -\left(\frac{\alpha_s}{4\pi}\right)^2 f^{ABC}\boldsymbol{T}_1^A\boldsymbol{T}_2^B\boldsymbol{T}_3^C\frac{\pi}{\varepsilon}\left(\Gamma_0 + \frac{2(1+\beta^2)}{\beta}\right)$$
$$\times\left[\Gamma_0\ln\left(\frac{-s_{31}}{-s_{32}}\right)L - 2\mathrm{Re}\left[c_{31}^{(1)}\right] + 2\mathrm{Re}\left[c_{32}^{(1)}\right]\right]$$

$$\left[\ln\boldsymbol{Z}_{\text{Tri}}^{(2)} + \ln\boldsymbol{Z}_{\text{Tri}}^{\dagger\,(2)}\right] = \left(\frac{\alpha_s}{4\pi}\right)^2 f^{ABC}\boldsymbol{T}_1^A\boldsymbol{T}_2^B\boldsymbol{T}_3^C\frac{2}{\varepsilon}\mathrm{Im}\left[g(\beta_{34})\right]\Gamma_0\left(\ln\frac{-\sigma_{41}v_4\cdot p_1}{-\sigma_{31}v_3\cdot p_1}\right)$$

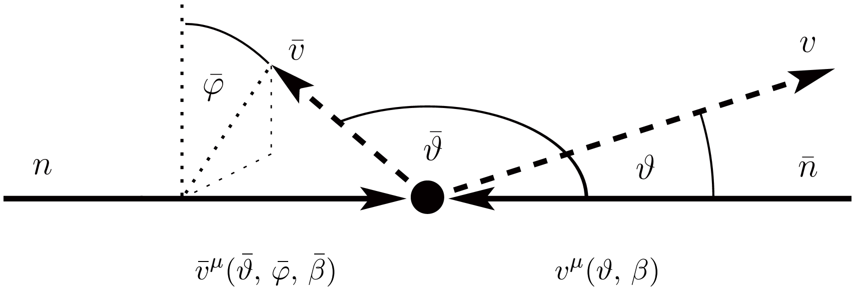
- There is a finite contribution to the renormalized soft function of the form

$$\frac{1}{2}\left[\boldsymbol{S_R}^{(1)},\,(\boldsymbol{Z}^{(1)}-\boldsymbol{Z}^{\dagger\,(1)})\right]_{\text{finite}} = -\left(\frac{\alpha_s}{4\pi}\right)^2 f^{abc}\boldsymbol{\mathcal{T}}_1^A\boldsymbol{\mathcal{T}}_2^B\boldsymbol{\mathcal{T}}_3^C\,\pi\left(\Gamma_0+\frac{2(1+\beta^2)}{\beta}\right)$$
$$\times\left\{\Gamma_0\ln\left(\frac{-s_{31}}{-s_{32}}\right)L^2-2\text{Re}\left[c_{31}^{(1)}\right]L+2\text{Re}\left[c_{32}^{(1)}\right]L\right.$$
$$\left.-2\text{Re}\left[c_{31}^{(1,\varepsilon)}\right]+2\text{Re}\left[c_{32}^{(1,\varepsilon)}\right]\right\}$$

- The Dipole structure at $\mathcal{O}(\alpha_s^2)$ originates only from the term

$$\left[\ln\boldsymbol{Z}^{(2)}+\ln\boldsymbol{Z}^{\dagger\,(2)}\right]$$

- Possible next steps include
  - ⇒ Calculate NNLO soft function for other $\text{SCET}_{\text{I/II}}$ observables
  - ⇒ Generalized kinematics for **non**-back-to-back case



- NNLO soft functions with massive particles in final state needed for extension of *GENEVA*
- Numerical calculation of bare soft function finished with good agreement to pole predictions
  - $\Rightarrow$ easy implementation of new kinematics/observables

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Fully automated NNLO soft functions with massive partons lie in the near future!

$\Rightarrow$ **Extension of *GENEVA* to processes with top quarks in final state**

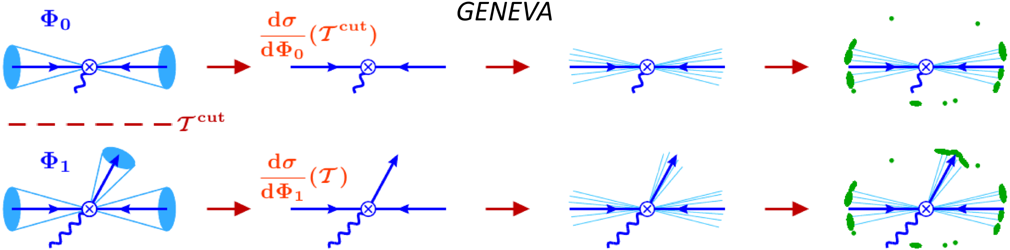
Thank You

Backup Slides

soft Wilson lines

$$\mathcal{S}_{v_i}(x) = P \exp\left[ig \int_{-\infty}^{0} \mathrm{d}s\, v_i \cdot A_s(x + s v_i)\right]$$

GENEVA



0-jettiness τ_0: Measure for how much the soft radiation is aligned to the beam directions

$$\min(k_+,\, k_-) = \min\left(k_T y_k,\, \frac{k_T}{y_k}\right)$$

Energy scales:

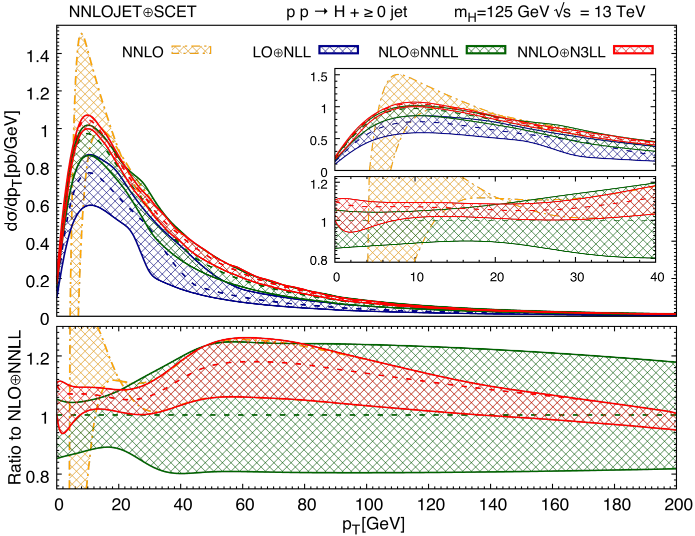
$$\Lambda_{\text{hard}} = Q \qquad \Lambda_{\text{col.}} = \sqrt{\tau}Q \qquad \Lambda_{\text{soft}} = \tau Q$$

Effective field theories

$$\mathcal{L}_{\text{EFT}} = \sum_i C_i(\mu) O_i(x)$$

- Wilson coefficients C_i depend on large energy scale Λ_{high} and matrix elements of sperators O_i depend on Λ_{low}
- Power counting of operators in expansion parameter $\lambda = \Lambda_{\text{low}}/\Lambda_{\text{high}}$
- Soft Collineaer Effective Theory (SCET) allows resummation of Sudakov logarithms at all orders of perturbation theory

Higgs p_T spectrum N3LL



[X. Chen, et al.; '18]

Light-cone coordinates and energy scales

Decomposition : $$q^\mu = \underbrace{(n\cdot q)}_{q_+}\frac{\bar{n}^\mu}{2} + \underbrace{(\bar{n}\cdot q)}_{q_-}\frac{n^\mu}{2} + q_\perp^\mu \equiv (\underbrace{+}_{n\cdot q}, \underbrace{-}_{\bar{n}\cdot q}, \perp)$$

Incoming partons (energetic, off-shell) with momenta p^μ and l^μ:

$$p^2 \sim l^2 \sim \lambda^2 Q^2$$

$$p^\mu \sim (\lambda^2,\, 1,\, \lambda)Q, \quad l^\mu \sim (1,\, \lambda^2,\, \lambda)Q$$

- There are four different modes in this process:

1. hard: $k^\mu \sim (1,\, 1,\, 1)Q$
2. soft: $k^\mu \sim (\lambda^2,\, \lambda^2,\, \lambda^2)Q$
3. collinear to p: $k^\mu \sim (\lambda^2,\, 1,\, \lambda)Q$
4. collinear to l: $k^\mu \sim (1,\, \lambda^2,\, \lambda)Q$

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