Higher-point energy correlators
made practical

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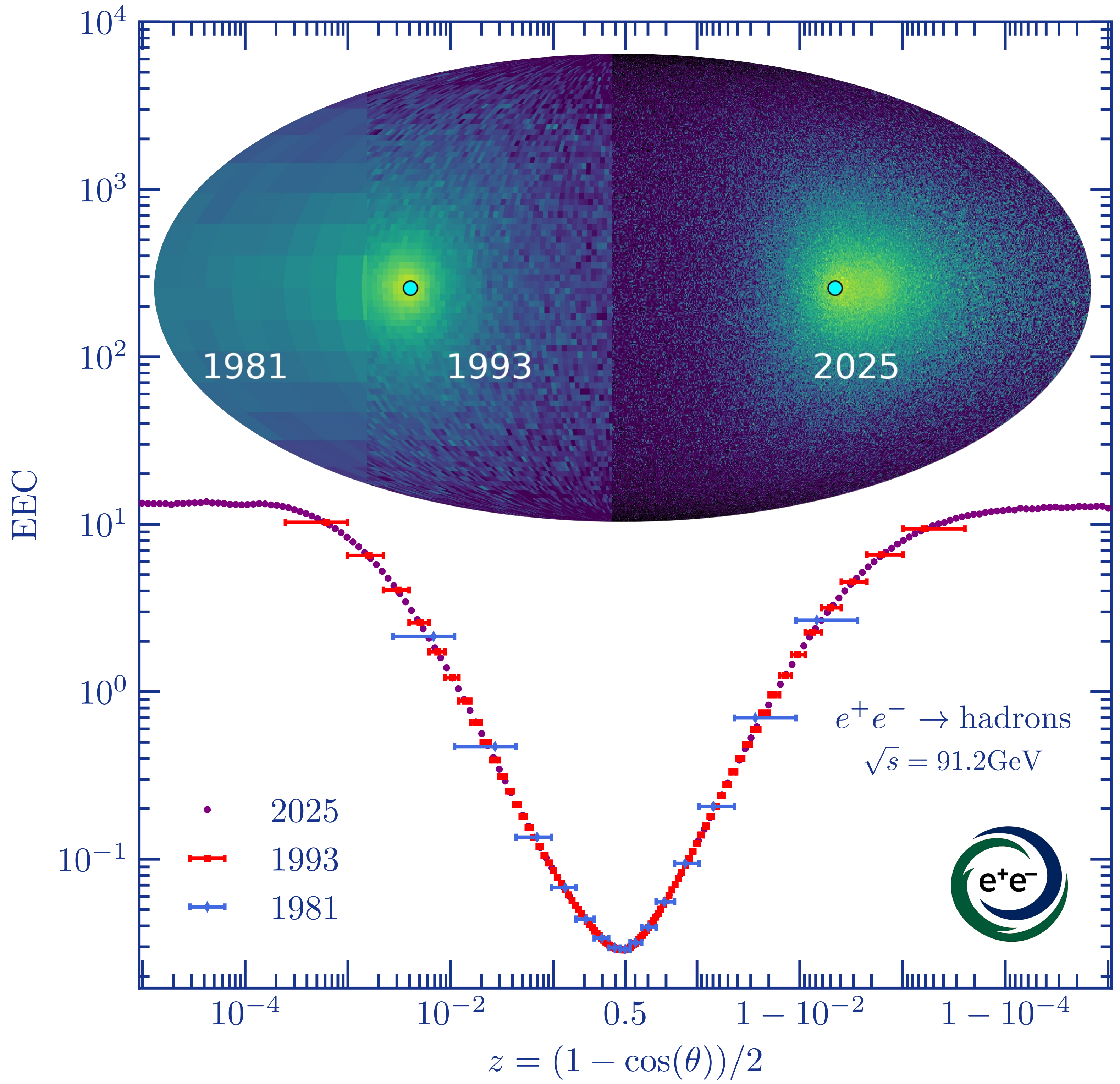
Outline

1. Why higher-point correlators matter
2. Making higher-point correlators practical
3. Factorization in back-to-back regime
4. Nonperturbative effects in collinear regime
5. Conclusions

Apologies: x, z, θ, R all used for angular distance

Why higher-point correlators matter

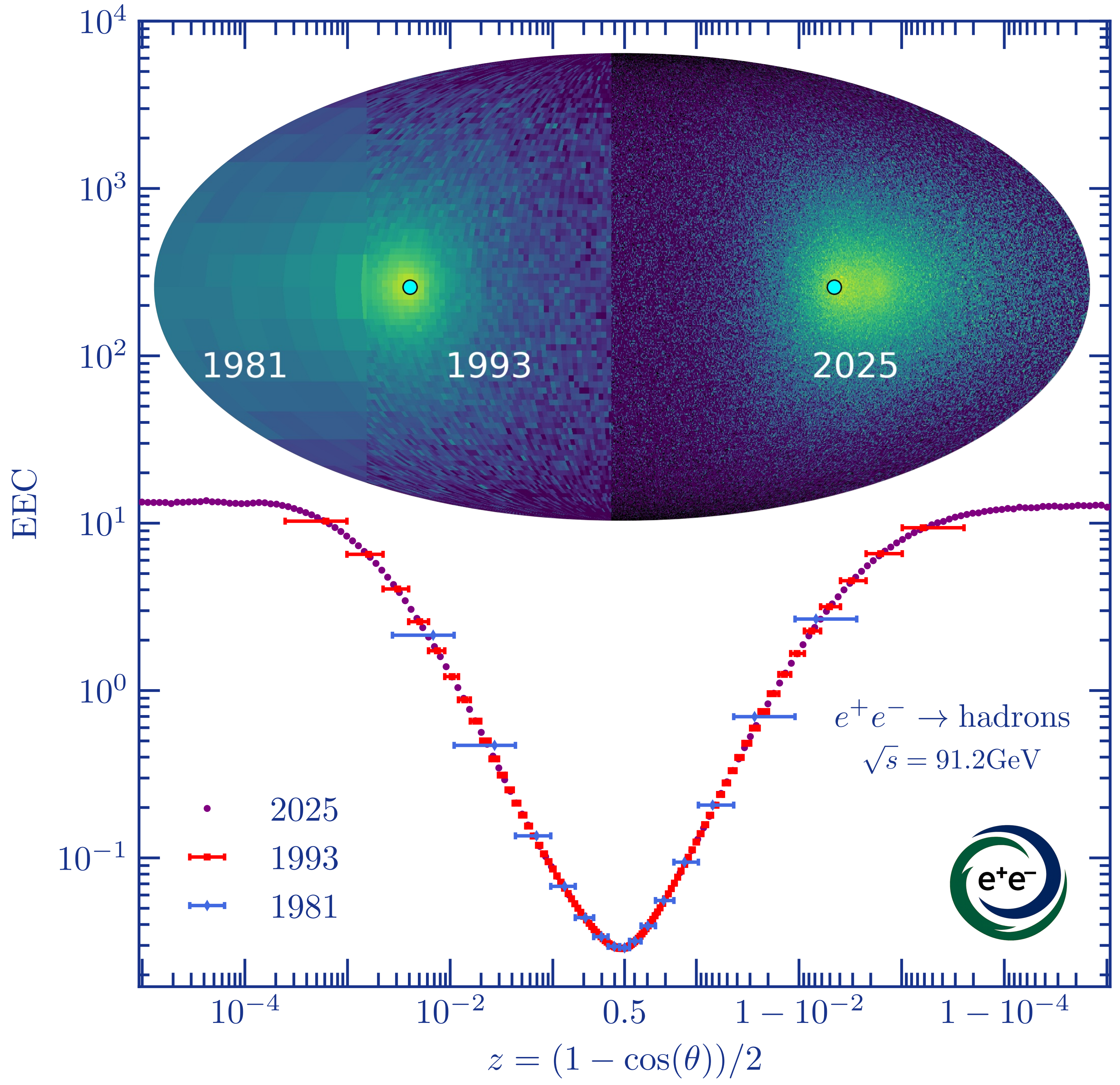
High precision energy-energy correlator

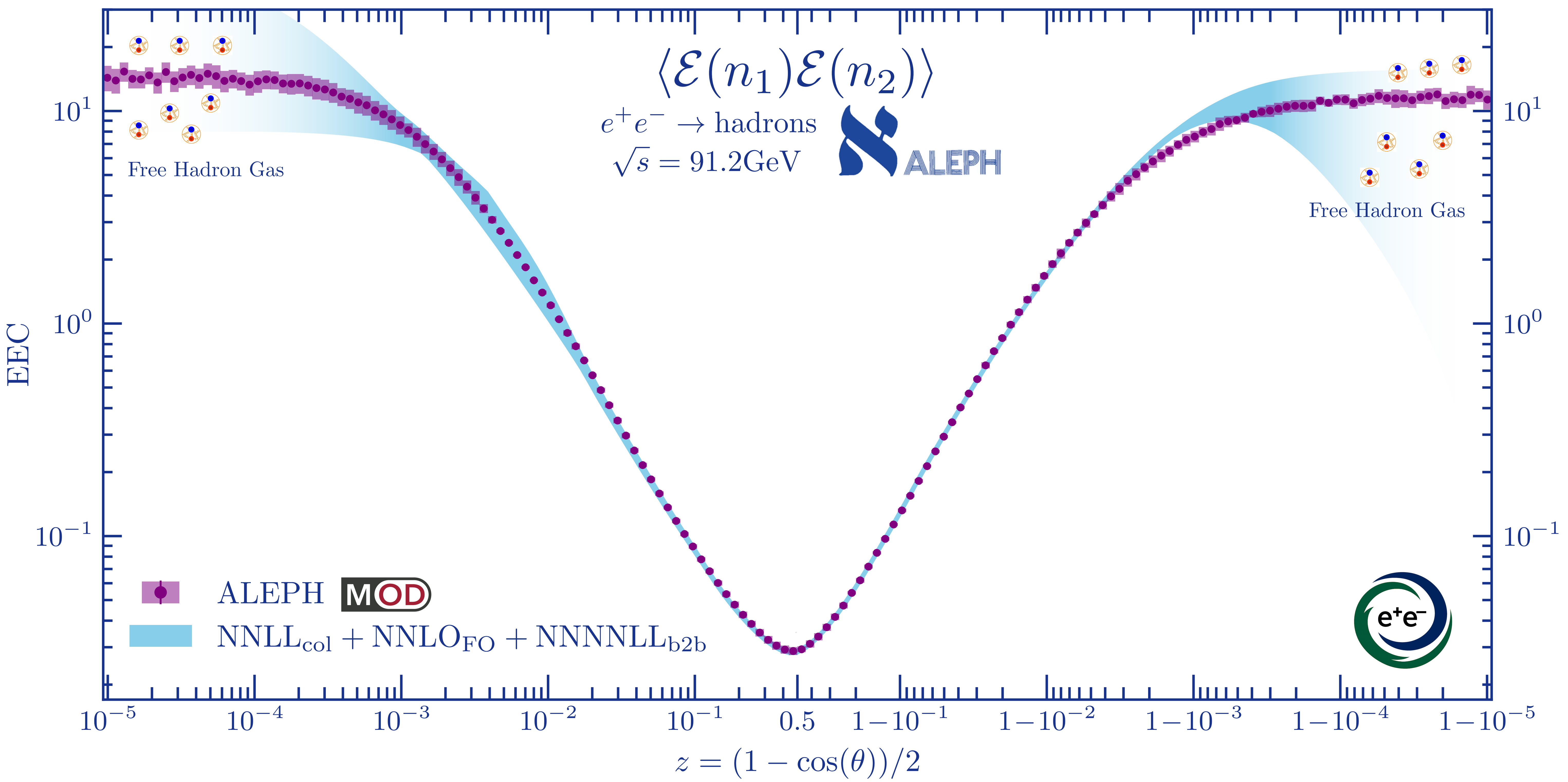


[Lee et al (theory: Jaarsma, Li, Moult, WW, Zhu)]

- Opportunity: charged particle tracks offer amazing angular resolution.

High precision energy-energy correlator

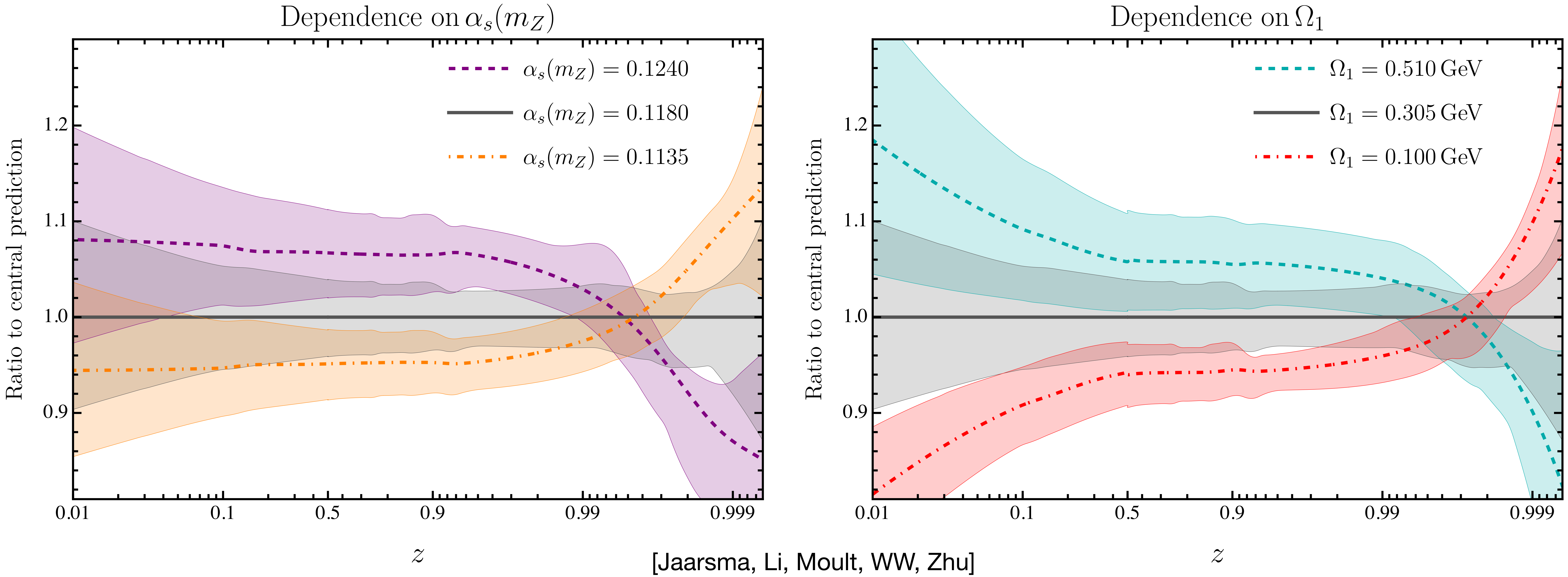




[Lee et al (theory: Jaarsma, Li, Moult, WW, Zhu)]

- Opportunity: charged particle tracks offer amazing angular resolution.
- **Precise** data & theory: NNLL collinear + N^{4}LL back-to-back.

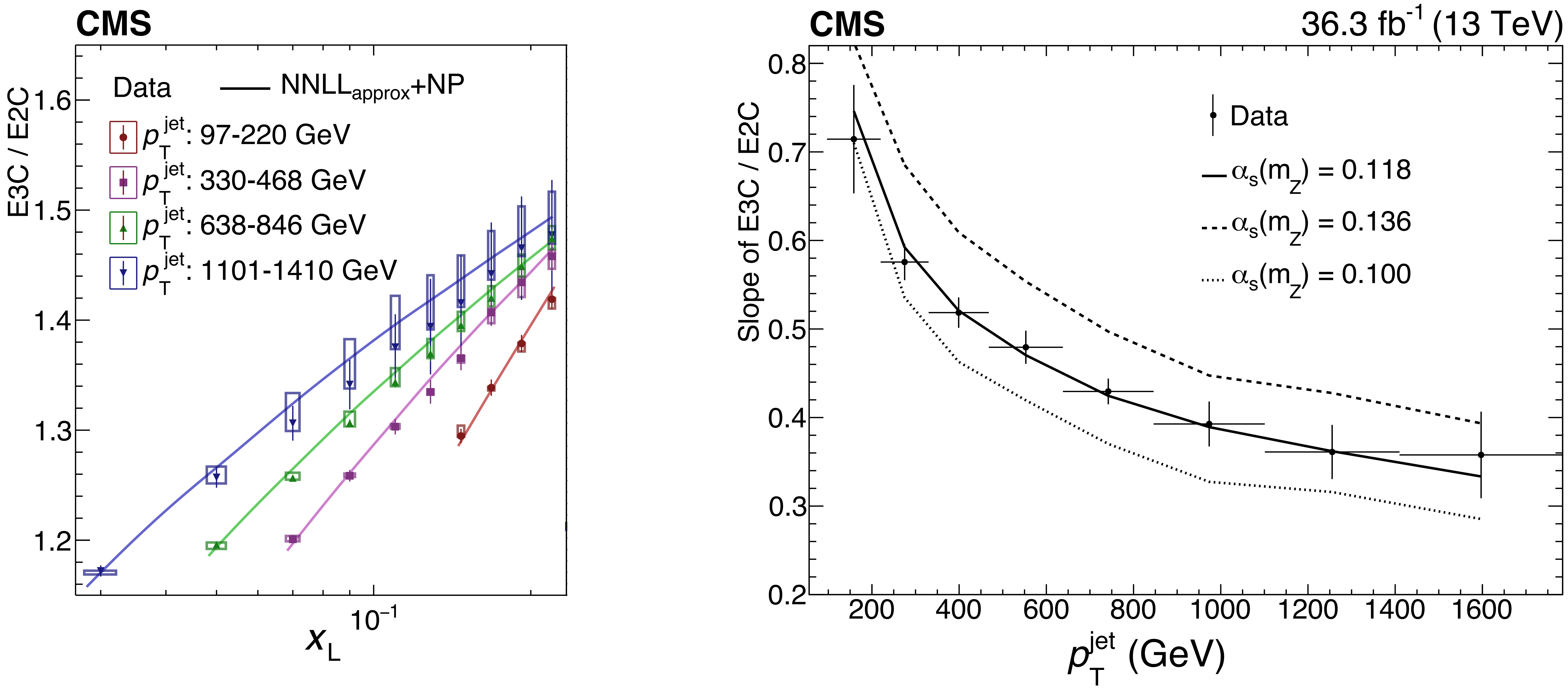
Break (α_s, Ω) degeneracy



[Jaarsma, Li, Moult, WW, Zhu]

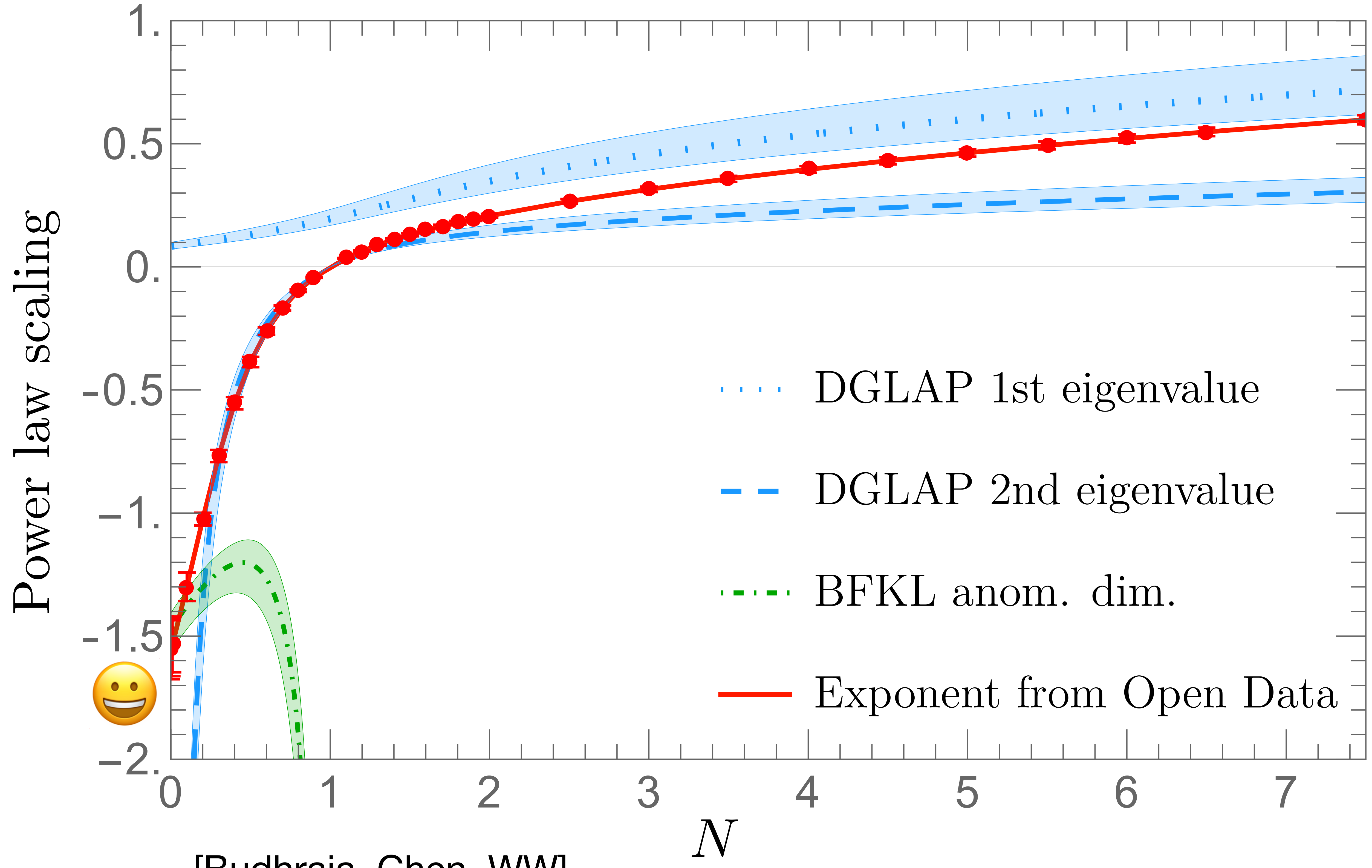
- Problem: α_s from event shapes correlated with leading nonperturbative effect.
- Potential solution: combine collinear and back-to-back region.

α_s from higher-point correlators



- Most precise α_s from jet substructure: extract from power law of E3C/EEC.

Small-x in jets from lower-point correlators

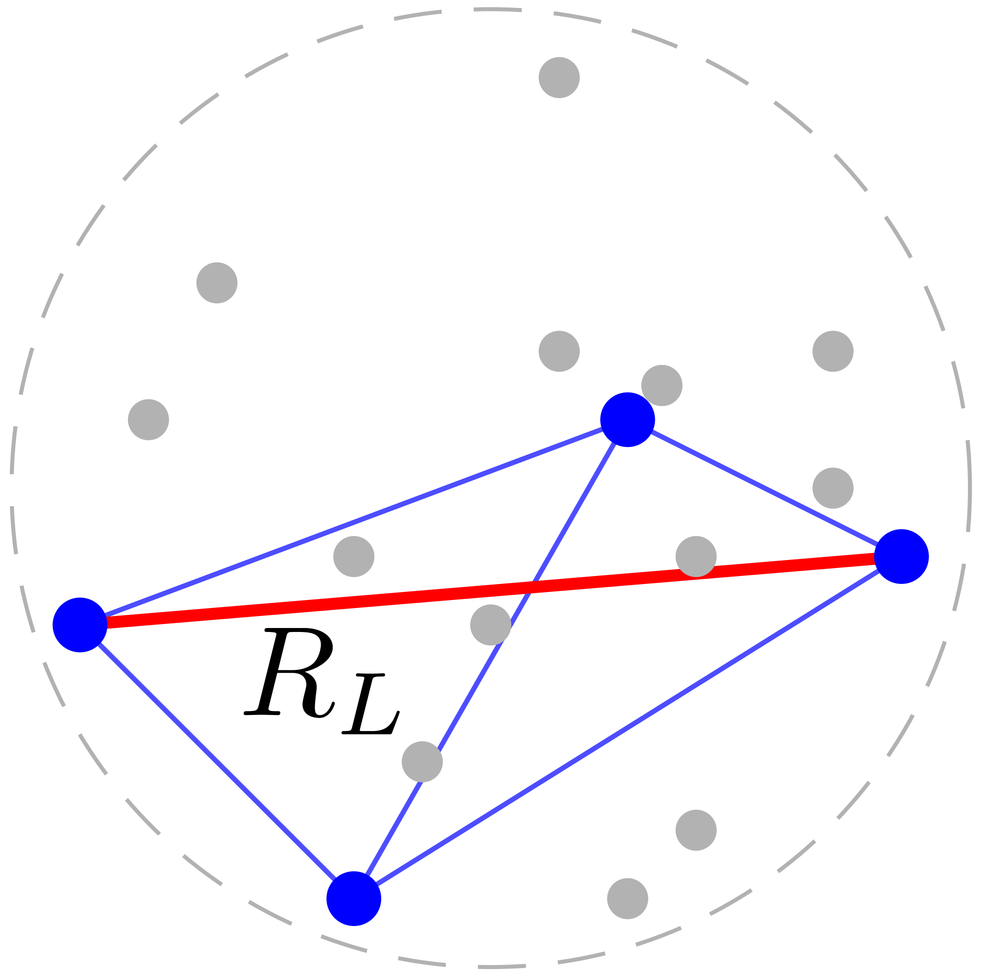


[Budhraja, Chen, WW]

- Analytically continue N-point energy correlator in N [Chen, Moult, Zhang, Zhu].
- $N \to 0$ corresponds to small-x, **tantalizing hint** in CMS open data.

Making higher-point correlators practical

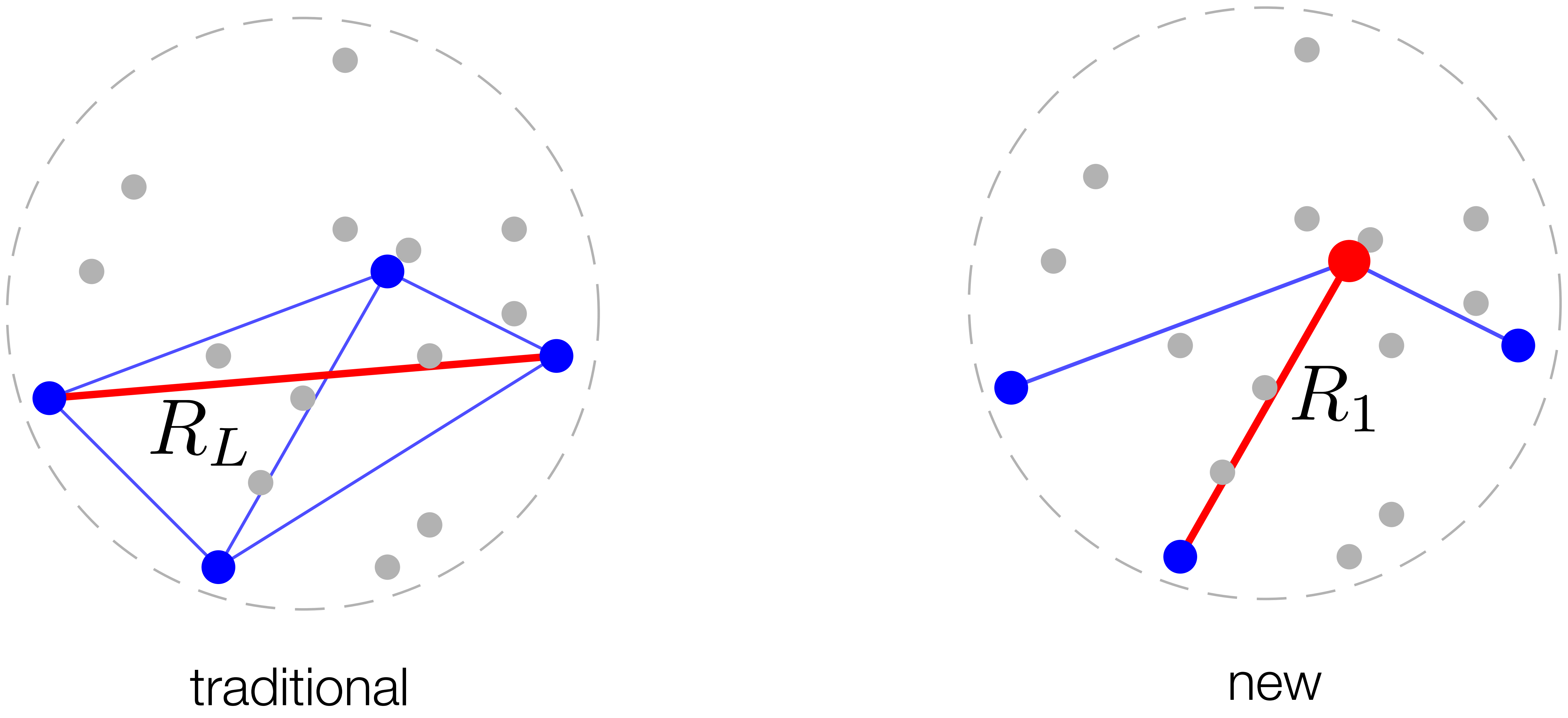
Fast N-point projected correlator



traditional

- Problem: computation time is $\mathcal{O}(M^N)$ for M particles, $\mathcal{O}(2^{2M})$ for non-integer N.

Fast N-point projected correlator



- Problem: computation time is $\mathcal{O}(M^N)$ for M particles, $\mathcal{O}(2^{2M})$ for non-integer N.

- Solution: isolate a special particle and only consider distances to it

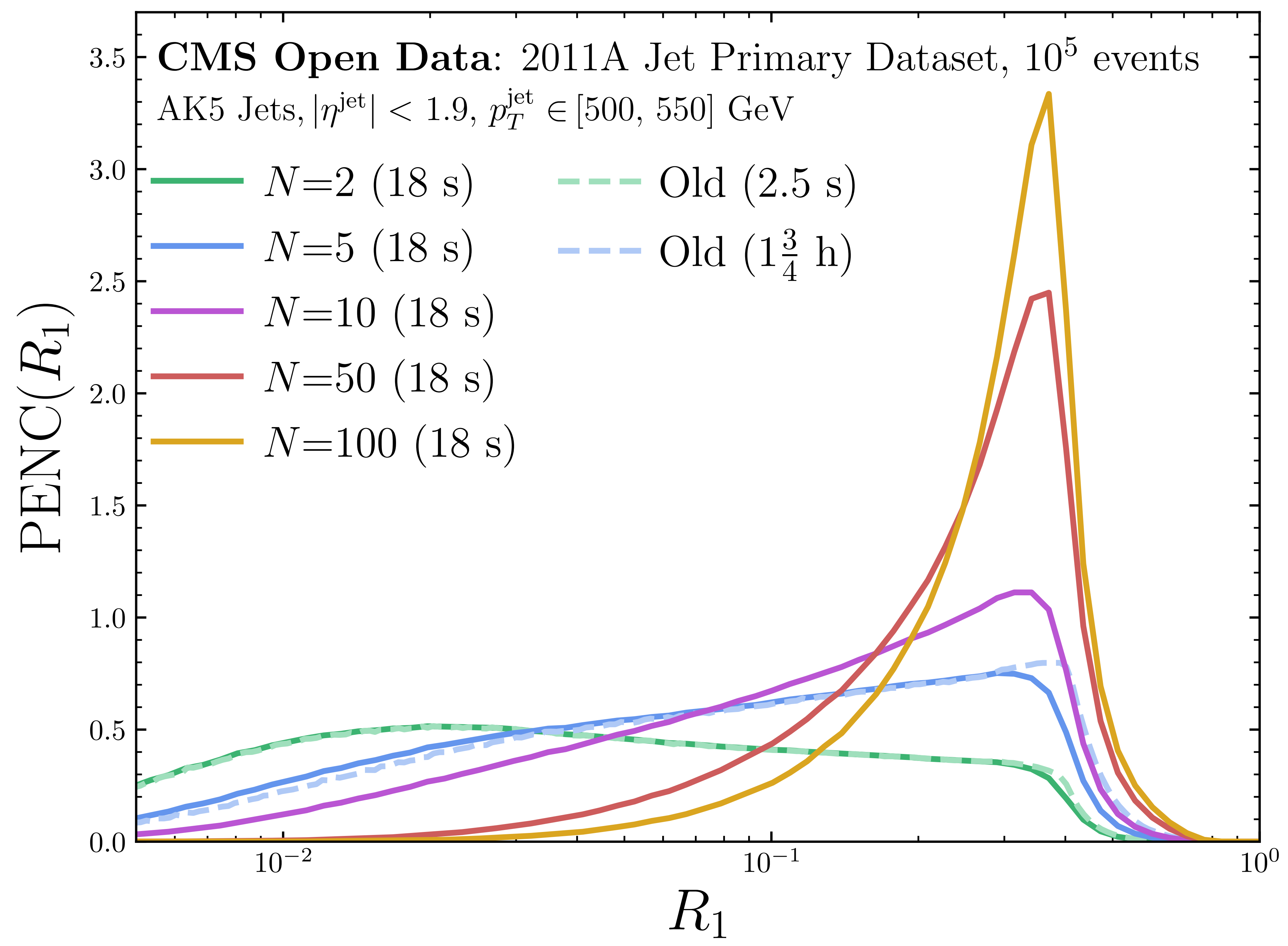
$$\frac{\mathrm{d}\sigma^{[N]}}{\mathrm{d}R_1} = \int \mathrm{d}\sigma \sum_{s} z_{s} \sum_{i_1, i_2, , \ldots} z_{i_1} z_{i_2} \cdots \delta(R_1 - \max\{R_{si_1}, R_{si_2}, \cdots\})$$

[Alipour-Fard, Budhraja, Thaler, WW]

Fast N-point projected correlator

- **Time is $\mathcal{O}(M^2 \ln M)$ for projected correlator for all N!**
- Follows from cumulative:

$$\int^{R_1} \mathrm{d}R_1 \, \frac{\mathrm{d}\sigma^{[N]}}{\mathrm{d}R_1'} = \int \mathrm{d}\sigma \sum_s z_s \, z_{\mathrm{disk}}(s, R_1)^{N-1}$$

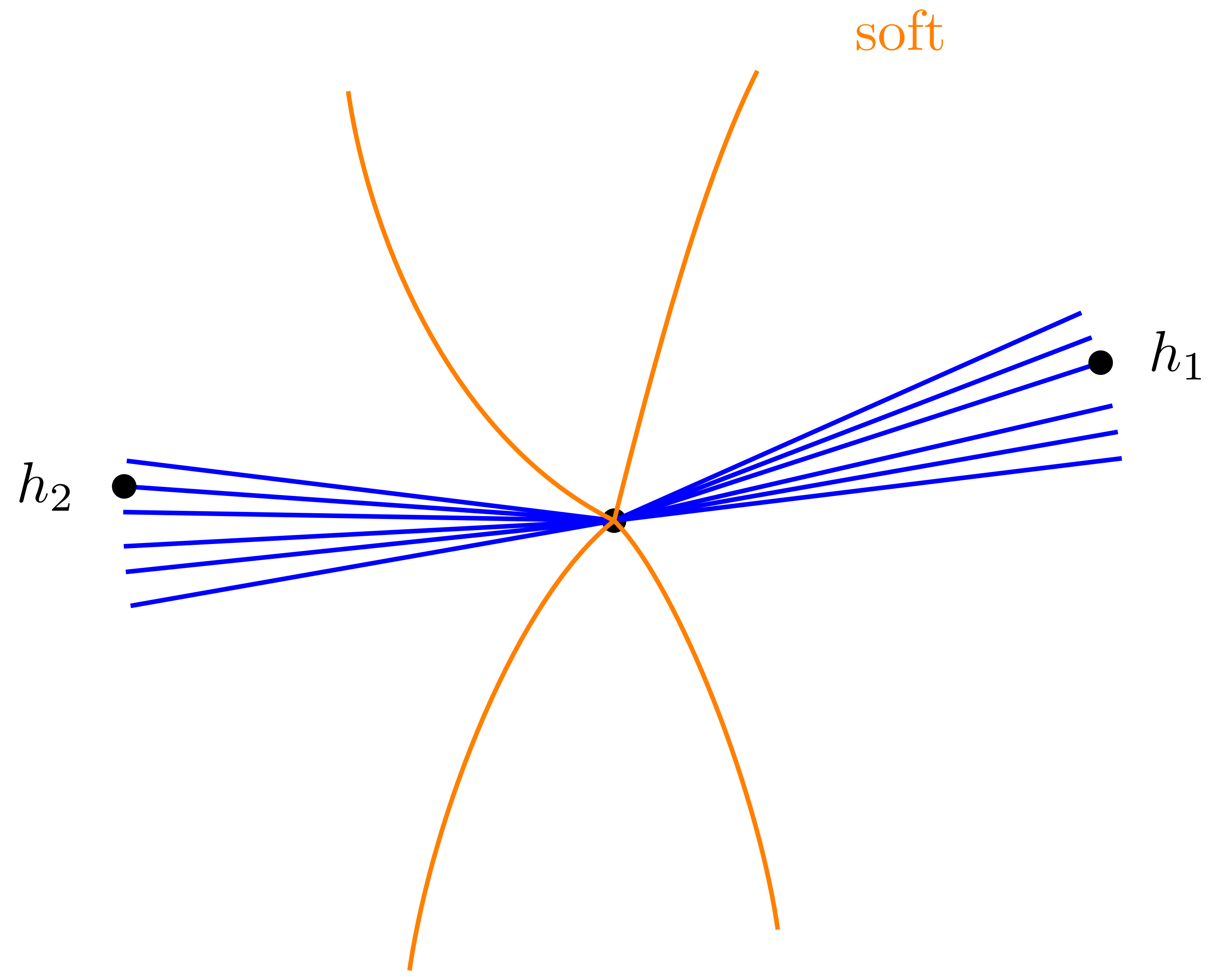


Factorization in back-to-back regime

Factorization for EEC in back-to-back regime

- Follows from TMD factorization for $e^+e^- \to 2$ hadrons [Moult, Zhu]

$$\frac{1}{\sigma_0}\frac{\mathrm{d}\sigma^{[N]}}{\mathrm{d}x} = H(Q;\mu)\int \mathrm{d}^2\mathbf{p}_n\, J(\mathbf{p}_n^2;\mu,\nu)\int \mathrm{d}^2\mathbf{p}_{\mathrm{soft}}\, S(\mathbf{p}_{\mathrm{soft}}^2;\mu,\nu)\int \mathrm{d}^2\mathbf{p}_{\bar{n}}\, J(\mathbf{p}_{\bar{n}}^2;\mu,\nu)$$
$$\times\,\delta\left[1 - x - \frac{1}{Q^2}\left(\mathbf{p}_n + \mathbf{p}_{\bar{n}} - \mathbf{p}_{\mathrm{soft}}\right)^2\right]$$



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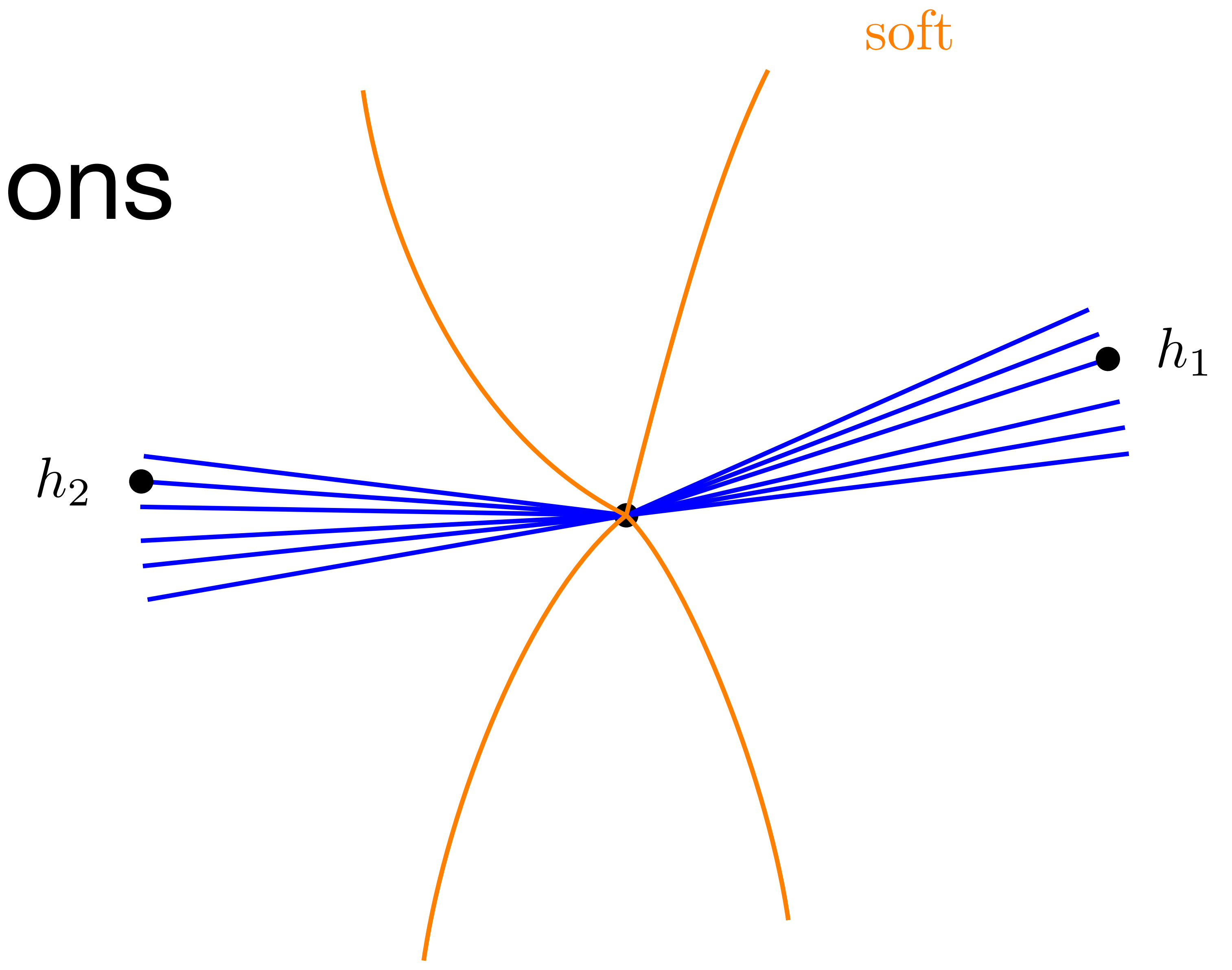
$$\times\,\delta\Big[1 - x - \frac{1}{Q^2}\big(\mathbf{p}_n + \mathbf{p}_{\bar n} - \mathbf{p}_{\mathrm{soft}}\big)^2\Big]$$

- Jet functions are integral of TMD fragmentation functions

$$J(\mathbf{p};\mu,\nu) = \sum_h \int \mathrm{d}z\, z\, D_{q\to h}(z,\mathbf{p};\mu,\nu)$$

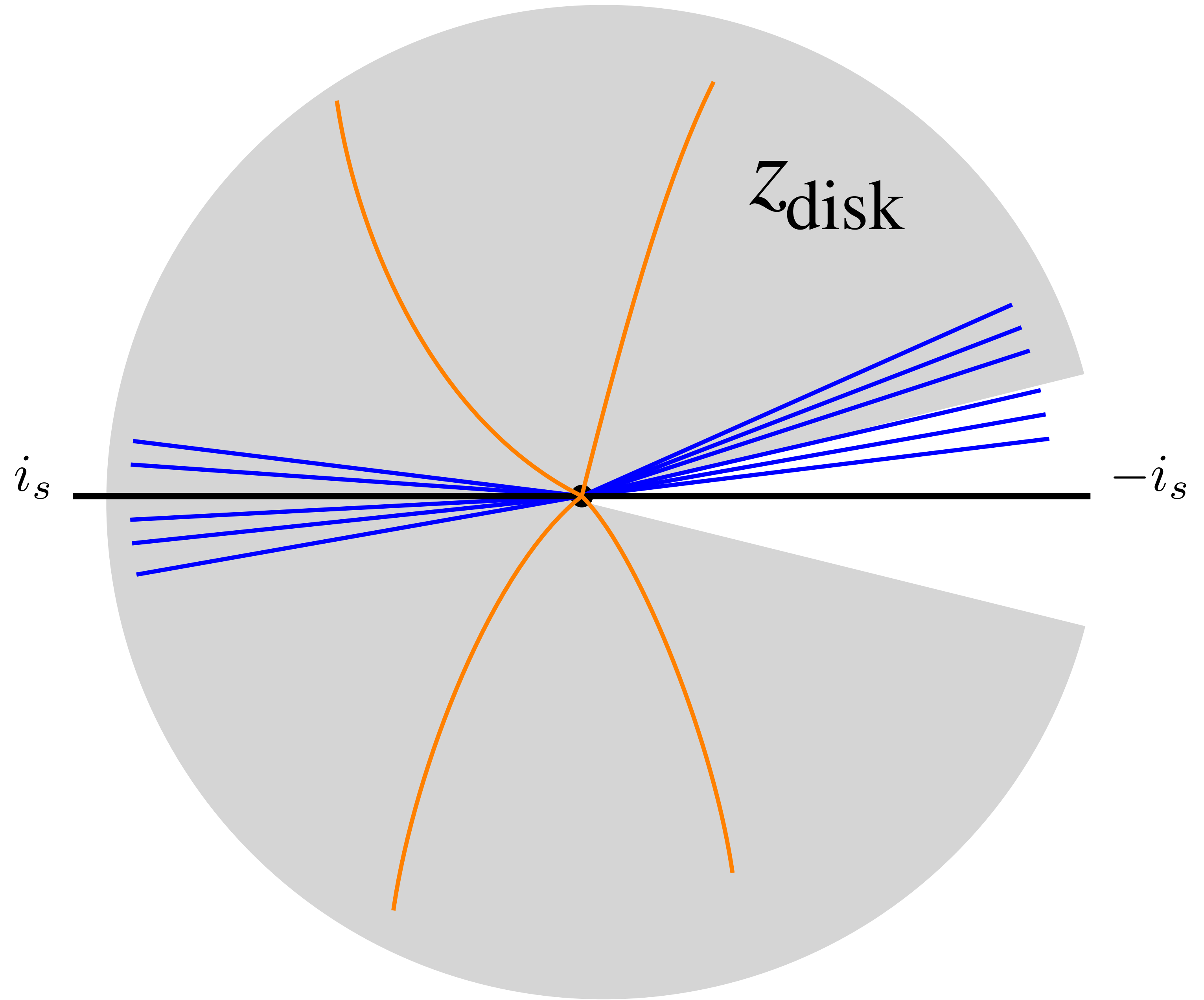


- Soft radiation not measured but provides recoil.

Factorization for N-point correlator in back-to-back

- **Summary: only one jet function changes.**
  - Hard function: same hard scattering.
  - Soft function: only provides recoil.
  - Jet function for jet with special particle: unchanged.
  - Other jet function depends on special particle

$$\boxed{\int^{R_1} \mathrm{d}R_1 \, \frac{\mathrm{d}\sigma^{[N]}}{\mathrm{d}R_1'} = \int \mathrm{d}\sigma \sum_{s} z_s \, z_{\mathrm{disk}}(s, R_1)^{N-1}}$$



Factorization for N-point correlator in back-to-back

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$$\times \int \mathrm{d}^2\mathbf{p}_{\bar{n}}\, J_{N-1}(\mathbf{p}_{\bar{n}}^2, \boxed{-\mathbf{p}_n + \mathbf{p}_{\mathrm{soft}}};\mu,\nu)$$
$$\times\, \delta\Big[1 - x - \frac{1}{Q^2}\big(\mathbf{p}_n + \mathbf{p}_{\bar{n}} - \mathbf{p}_{\mathrm{soft}}\big)^2\Big] + (n \leftrightarrow \bar{n})$$

- Other jet function depends on special particle through distance to antipode

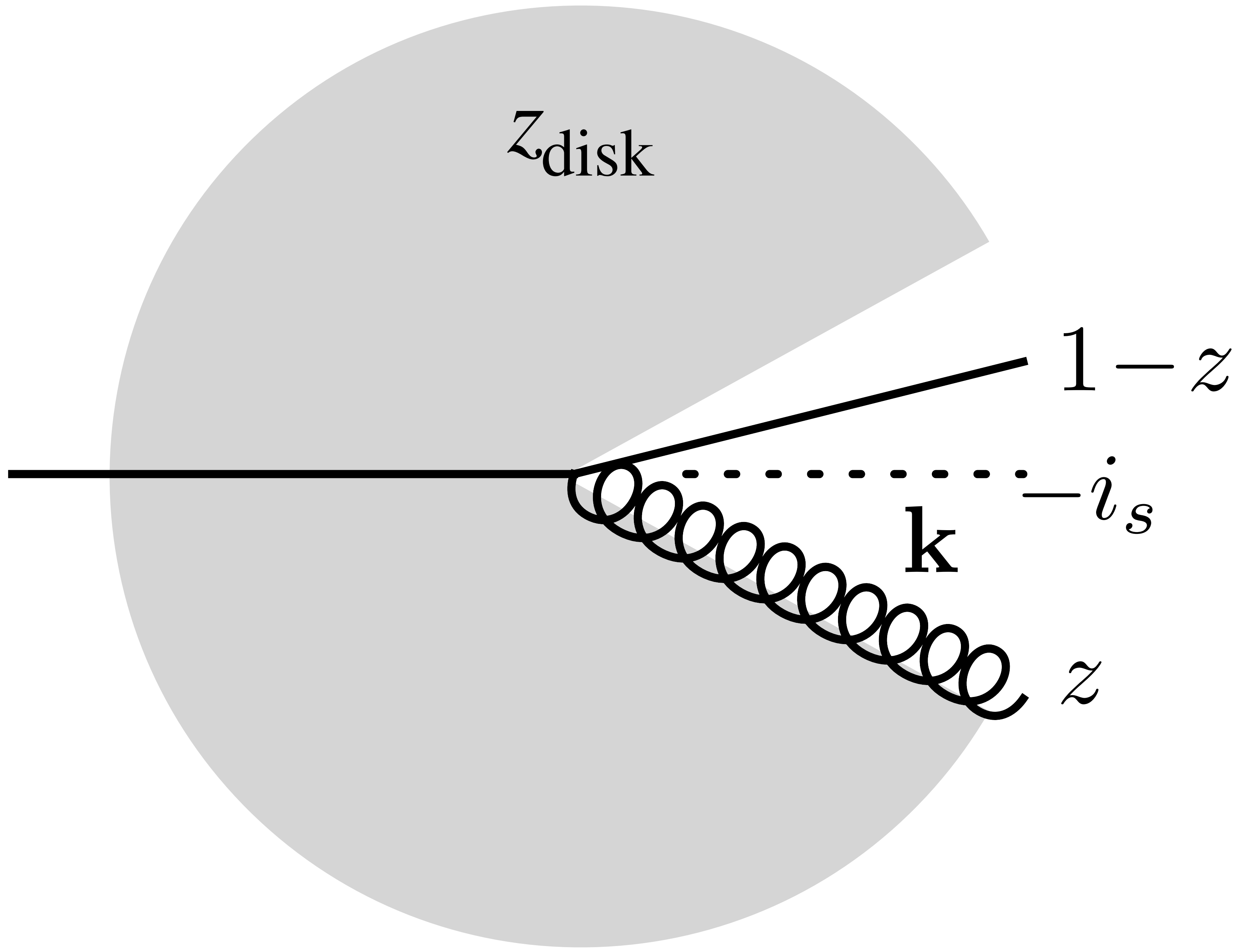
Jet function

- At LO, the new jet function is trivial

$$J_{N-1}(\mathbf{p}, \mathbf{p}_{\rm ap}; \mu, \nu) = (2^{N-1} - 1^{N-1})\, \delta^2(\mathbf{p})$$

both jets in z_{disk} only jet with special particle

Jet function



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- At NLO, without recoil ($\mathbf{p}_{\text{ap}} = 0$), the measurement is:

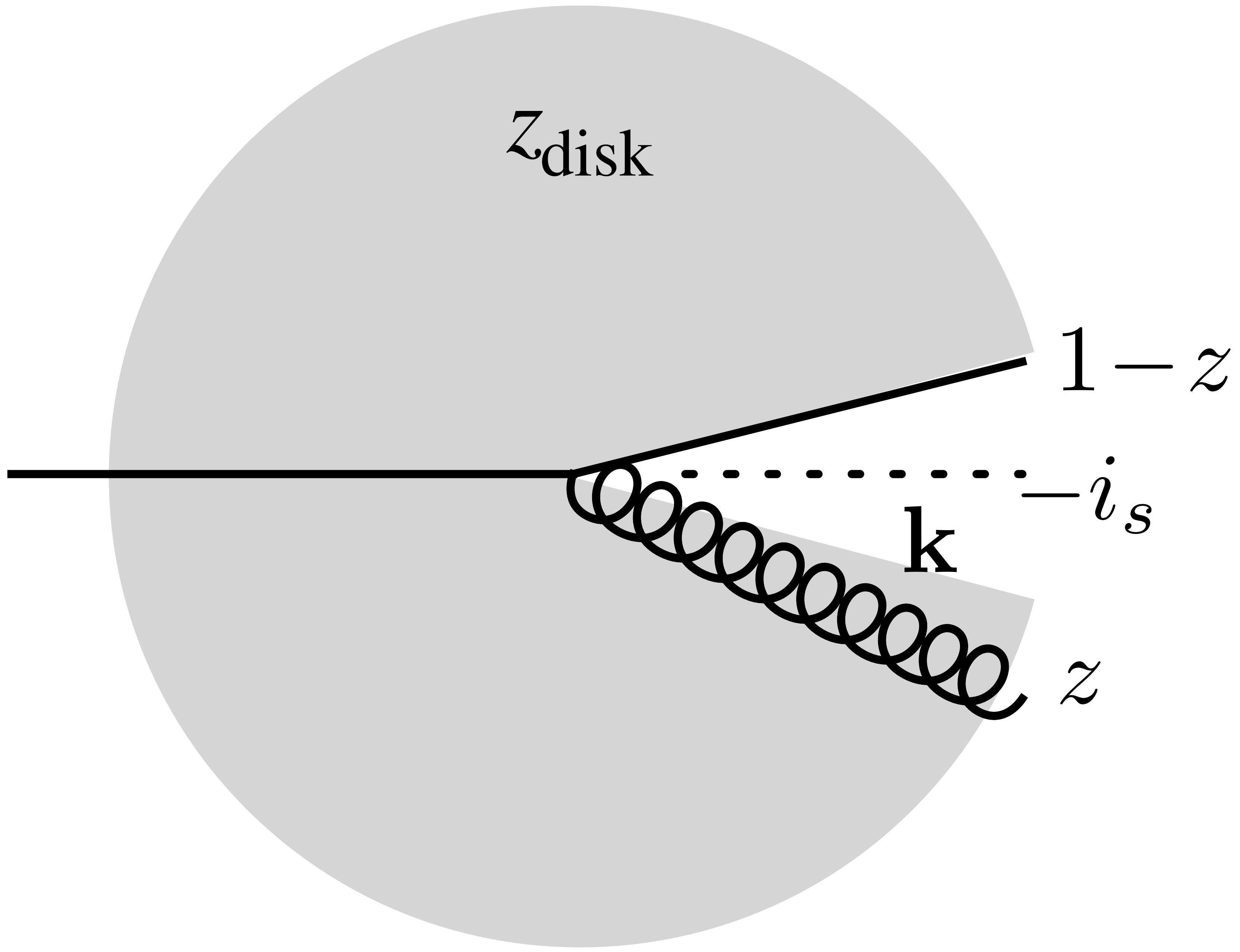
$$\Theta\Big(z < \frac{1}{2}\Big)\Big\{[(1+z)^{N-1} - 1^{N-1}]\, \delta^2\Big(\mathbf{p} - \frac{\mathbf{k}}{z}\Big) +$$

difference of z_{disk}^{N-1}

"angle" of particle with z

particle with z furthest from antipode

Jet function



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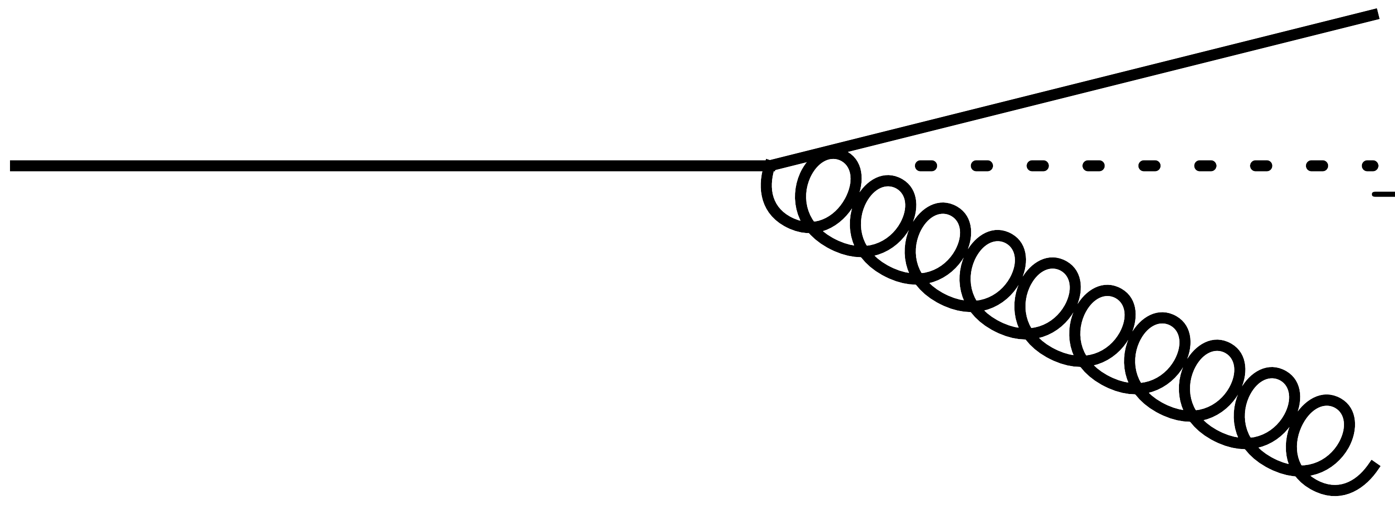
difference of z_{disk}^{N-1}

"angle" of particle with z

particle with z furthest from antipode

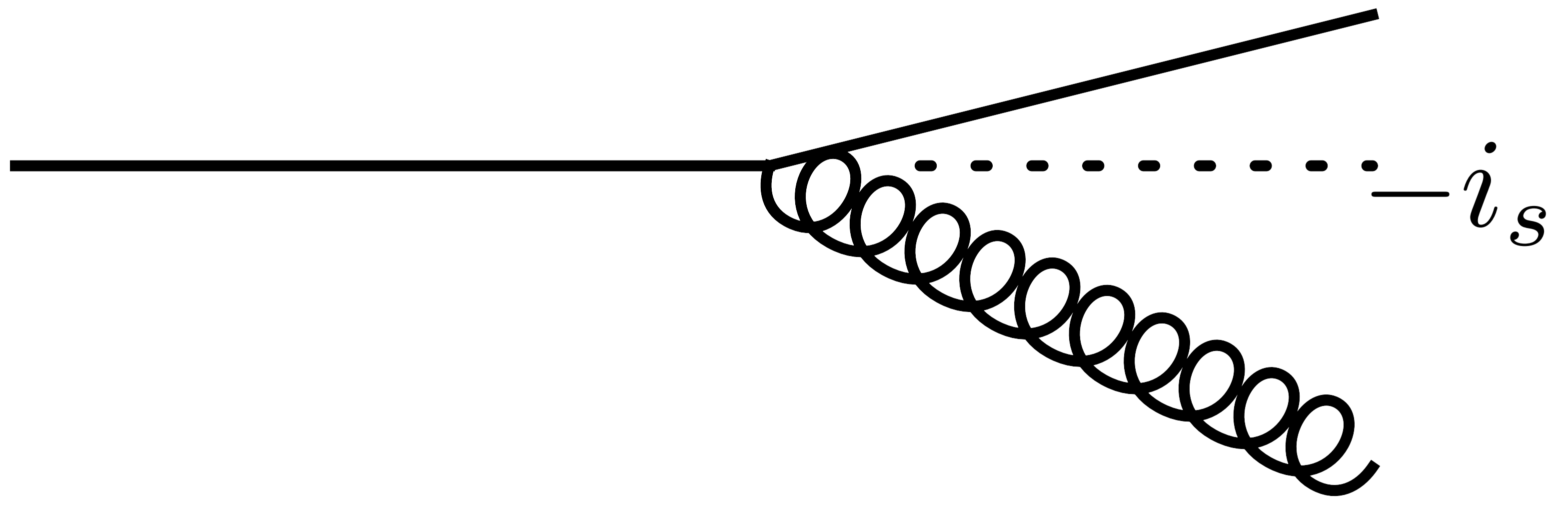
Jet function



- Jet function **without recoil**:

$$\frac{J_{N-1}(\mathbf{p},\mathbf{0};\mu,\nu)}{2^{N-1}-1} = \delta^2(\mathbf{p}) + \frac{\alpha_s C_F}{\pi}\left\{ j_{N-1}\delta^2(\mathbf{p}) + \left[2\ln\left(\frac{Q}{\nu}\right) - \frac{3}{2}\right]\frac{1}{2\pi\mu^2}\left[\frac{1}{\mathbf{p}^2/\mu^2}\right]_+ \right\}$$

Jet function



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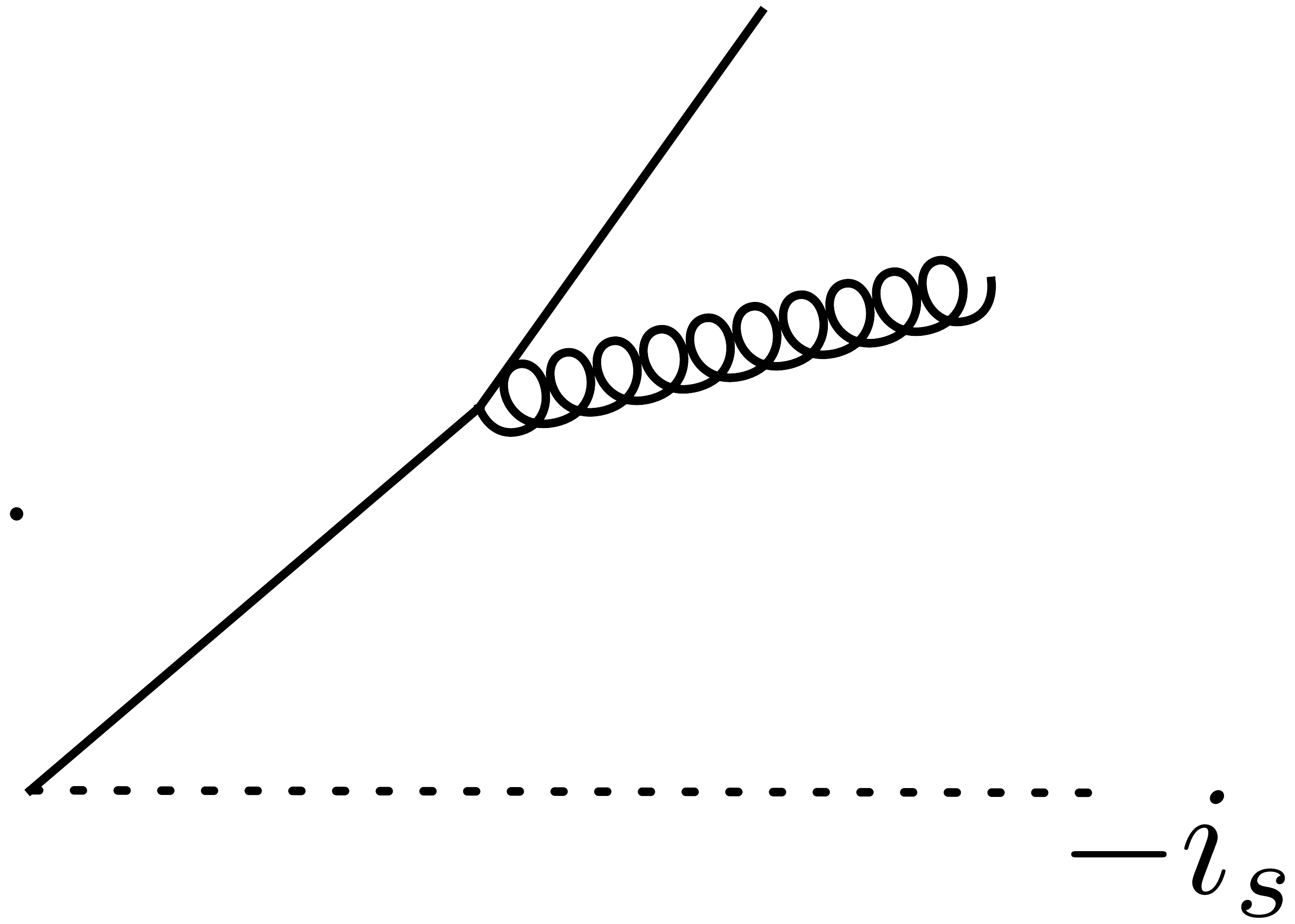
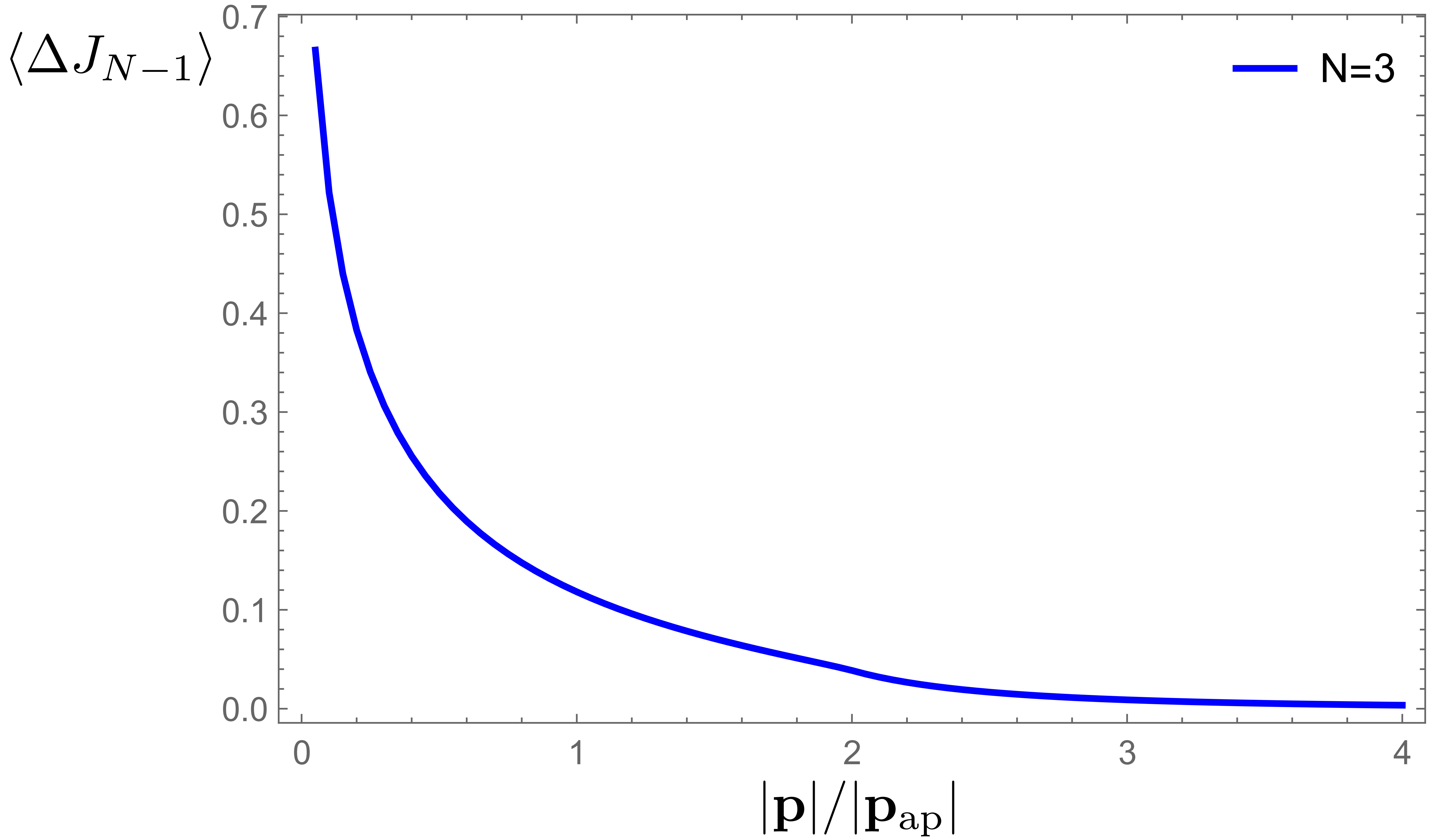
- Jet function in **infinite recoil limit**:
    - Boundary $\phi = \frac{\pi}{2}$ instead of $z = \frac{1}{2}$
    - Only modifies constant:

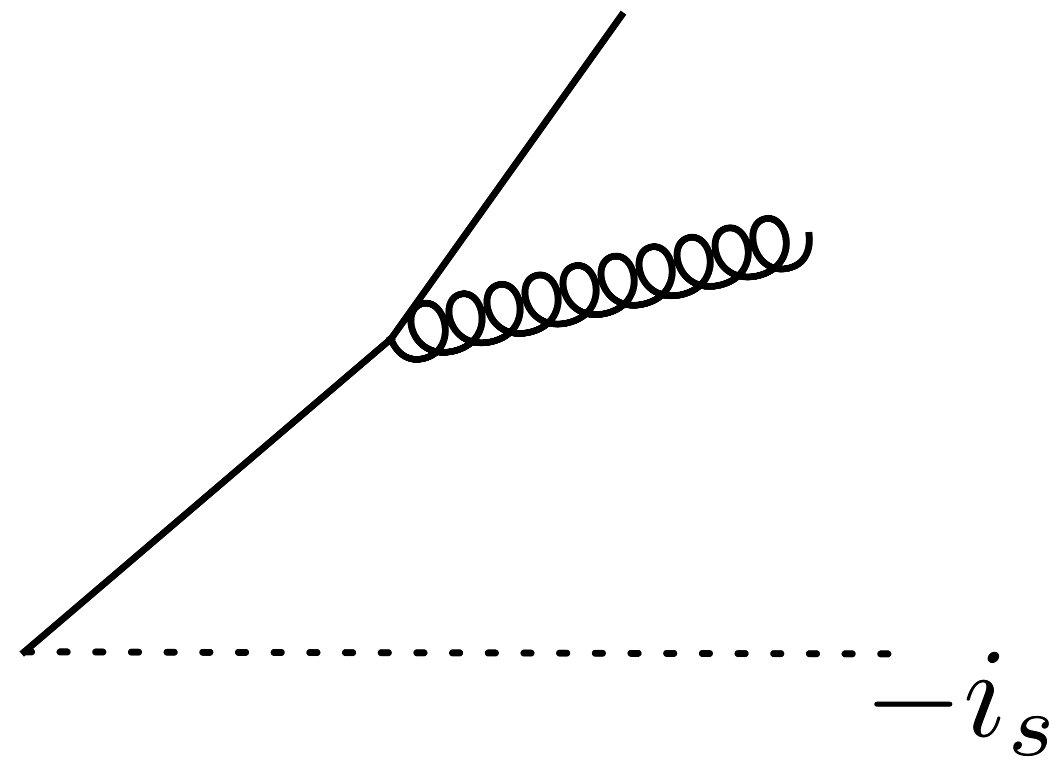
$$\frac{\Delta J_{N-1}}{2^{N-1}-1} \overset{|\mathbf{p}|\to 0}{=} \frac{\alpha_s C_F}{\pi}\, j'_{N-1}\,\delta^2(\mathbf{p})\,, \qquad j'_2 = -\frac{1}{24} - \frac{1}{2}\ln 2\,.$$



Jet function with recoil

- Effect of recoil is finite correction: $\Delta J_{N-1}(|\mathbf{p}|/|\mathbf{p}_{\mathrm{ap}}|,\, \phi)$

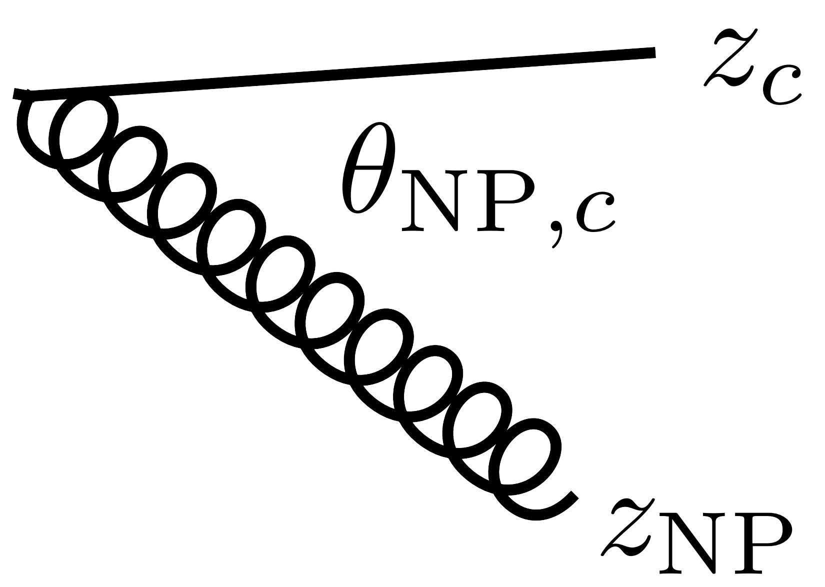




✓ Checked against large recoil limit.

Nonperturbative effects in collinear regime

Leading nonperturbative effects



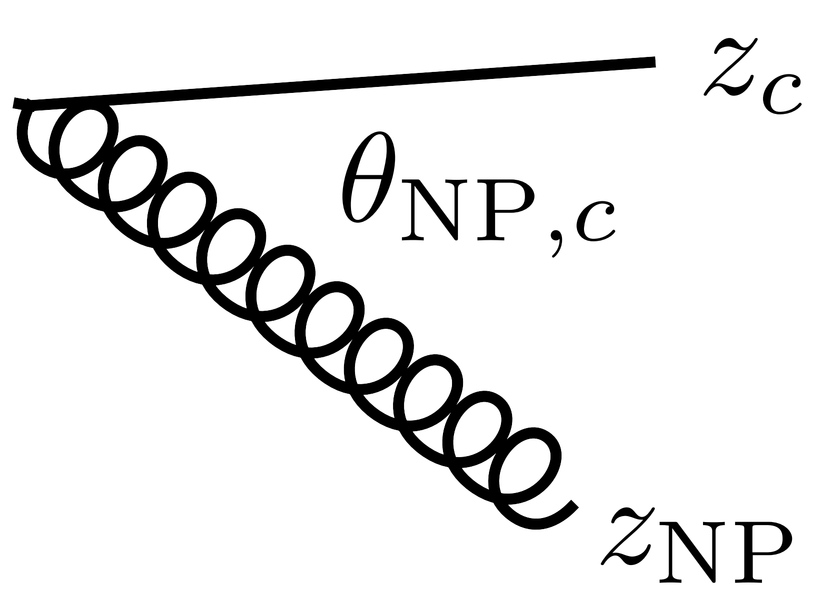
- Contribution to energy correlator from collinear and nonperturbative particle:

collinear is special particle

nonperturbative is special

$$\left\{z_c\left[\left(z_c+z_{\mathrm{NP}}\right)^{N-1}-z_c^{N-1}\right]+z_{\mathrm{NP}}\left[\left(z_c+z_{\mathrm{NP}}\right)^{N-1}-z_{\mathrm{NP}}^{N-1}\right]\right\}\delta(\theta-\theta_{\mathrm{NP},c})$$

Leading nonperturbative effects



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$$\left\{z_c\left[(z_c + z_{\mathrm{NP}})^{N-1} - z_c^{N-1}\right] + z_{\mathrm{NP}}\left[(z_c + z_{\mathrm{NP}})^{N-1} - z_{\mathrm{NP}}^{N-1}\right]\right\}\delta(\theta - \theta_{\mathrm{NP},c})$$
$$\approx \left[N z_c^{N-1} z_{\mathrm{NP}} - z_{\mathrm{NP}}^{N}\right]\delta(\theta - \theta_{\mathrm{NP},c})$$

  - First term yields familiar result for $N \geq 2$ [Lee, Pathak, Stewart, Sun; Chen, Monni, Xu, Zhu].
  - Second term dominates for $N < 1$.

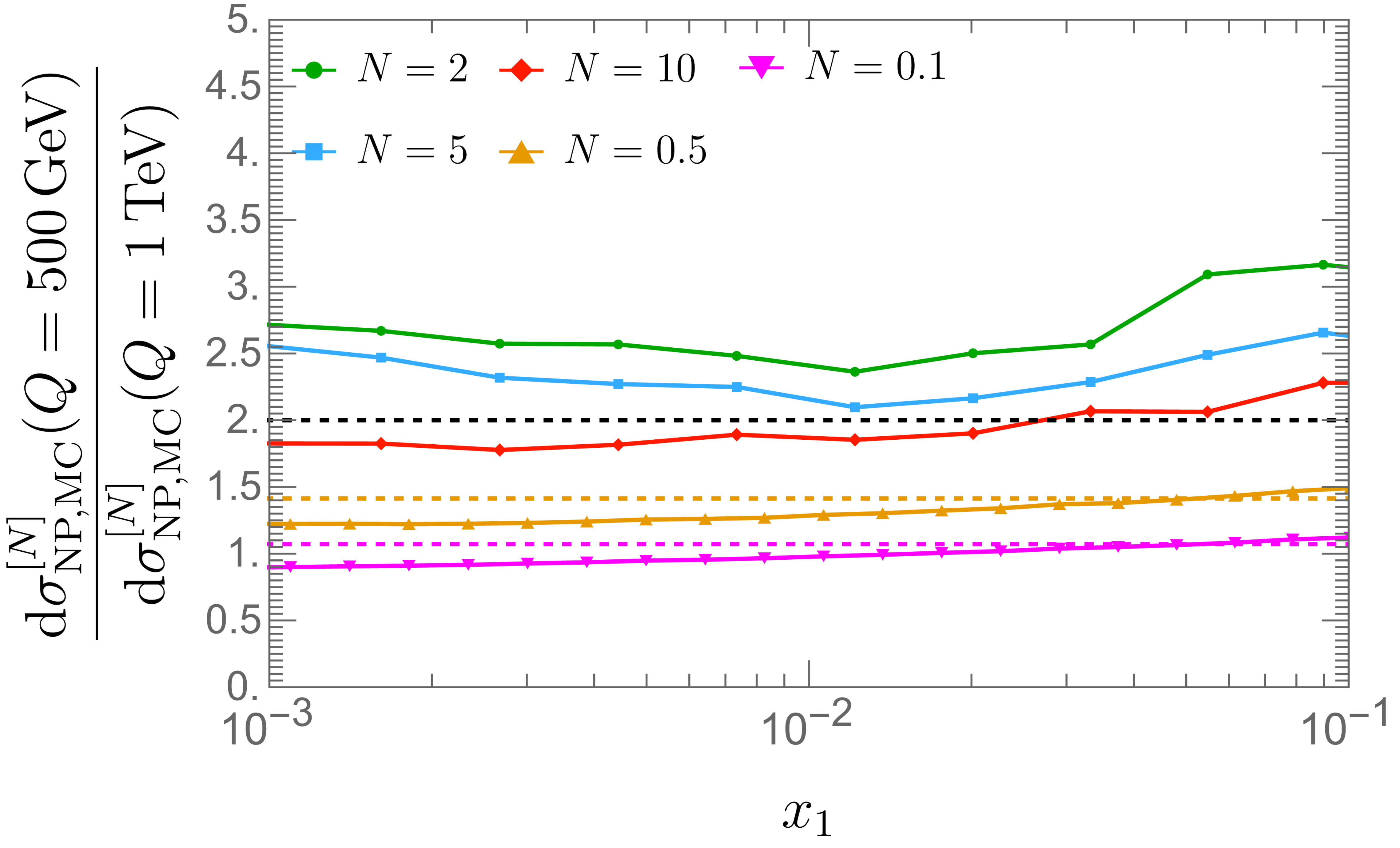
Leading nonperturbative effects

- Leading nonperturbative corrections to cross section in collinear region:

$$\boxed{\frac{1}{\sigma}\frac{\mathrm{d}\sigma_{\mathrm{NP}}}{\mathrm{d}x_1} = \frac{N\Omega_1}{2^N Q x_1^{3/2}} + \frac{\Omega^{[N]}}{2^N Q^N x_1^{1+N/2}}}$$

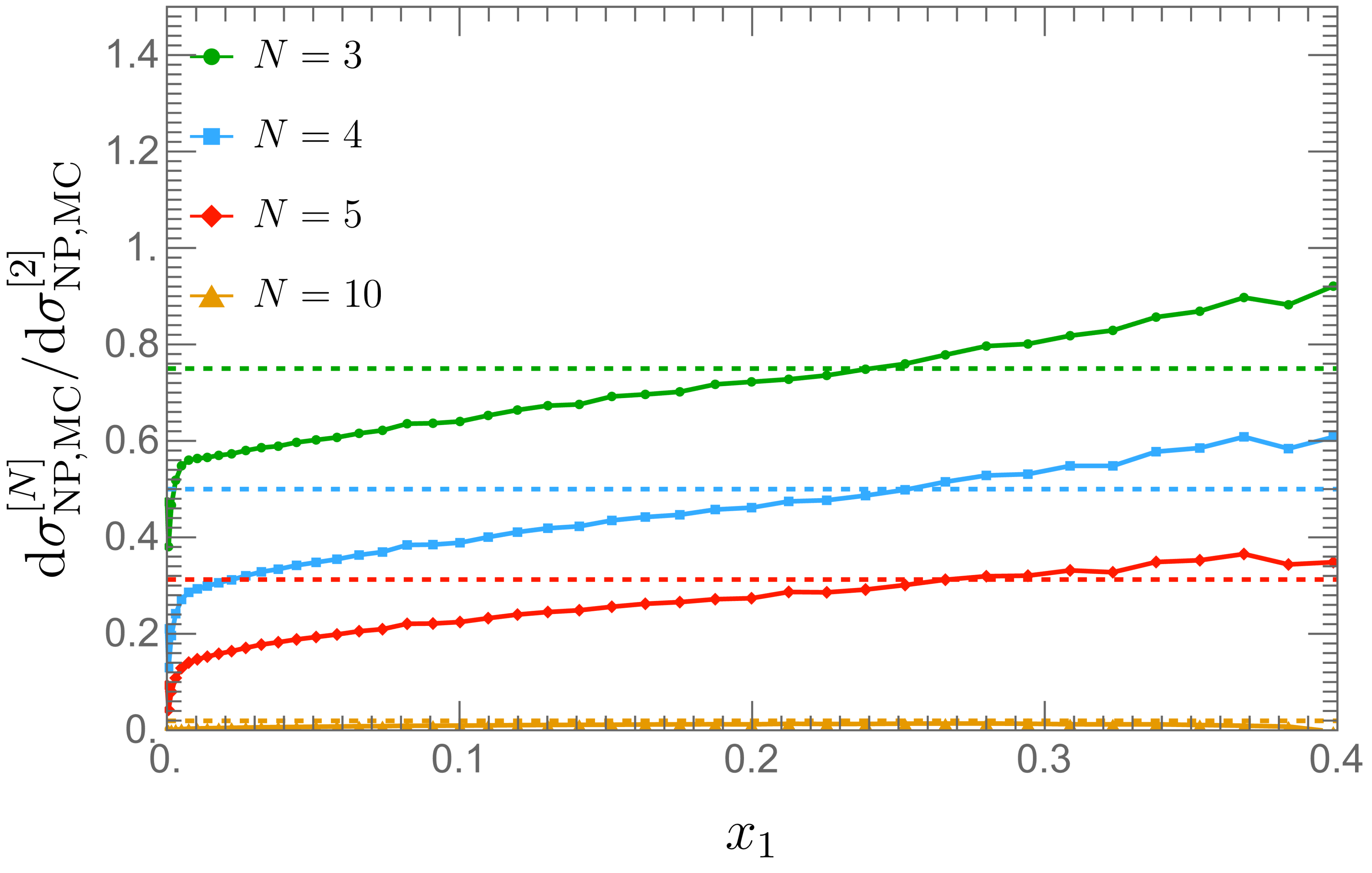
- Features of second term:
  - Classical scaling modified $1/x_1^{3/2} \to 1/x_1^{1+N/2}$.
  - Expect $\Omega^{[N]} \approx (\Omega_1)^N$, though $\langle z_{\mathrm{NP}}^N \rangle \neq \langle z_{\mathrm{NP}} \rangle^N$ in general.
- Study in Pythia: taking the difference between hadronization on/off.

Dependence on Q



- For $N \geq 2$, the $1/Q$ scaling is not so clear in Pythia.
- For $N < 1$, the $1/Q^N$ scaling is clearly present.

Dependence on N for $N \geq 2$



- Normalization is correct, but power-law in x_1 also changes.

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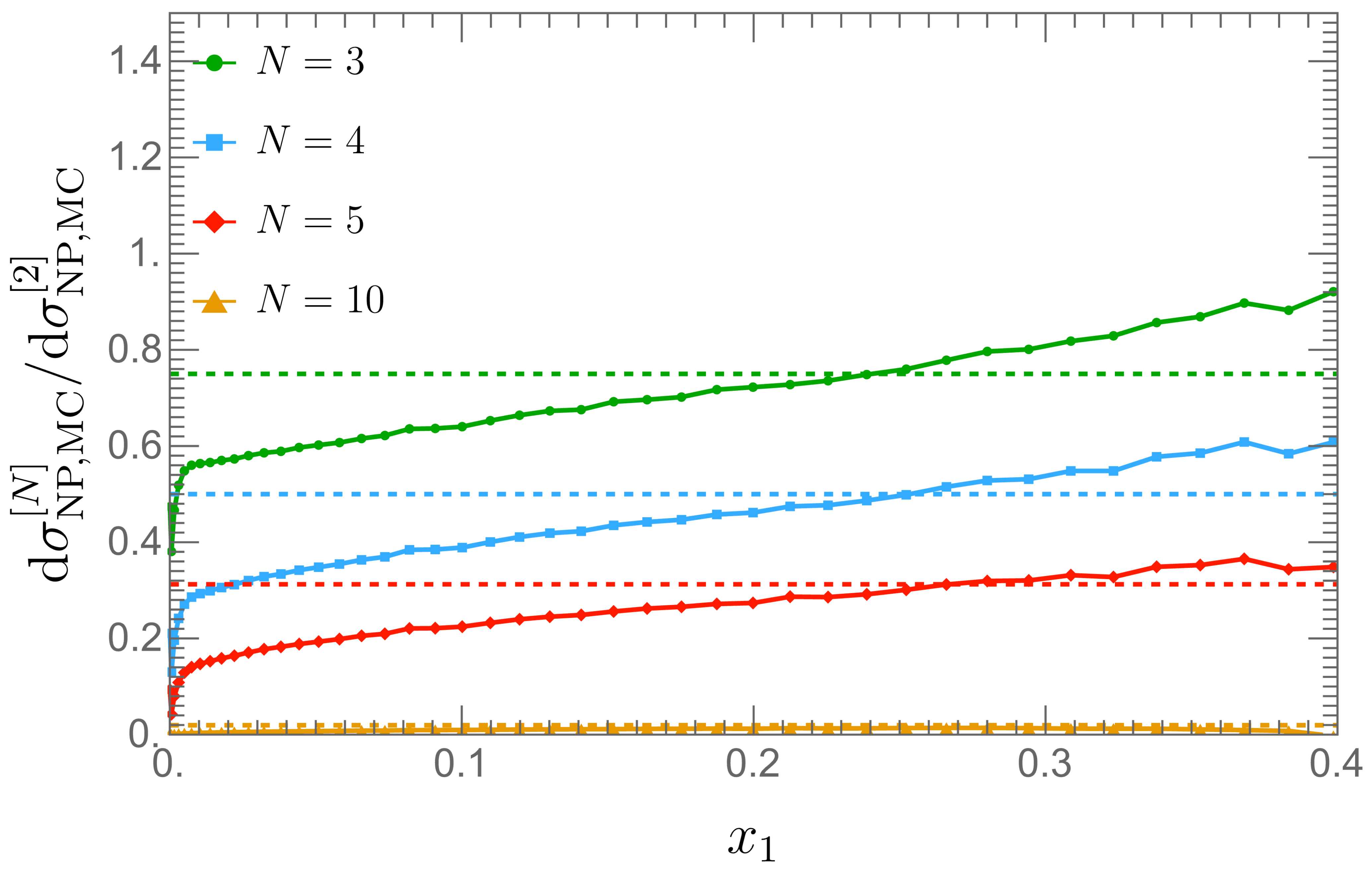
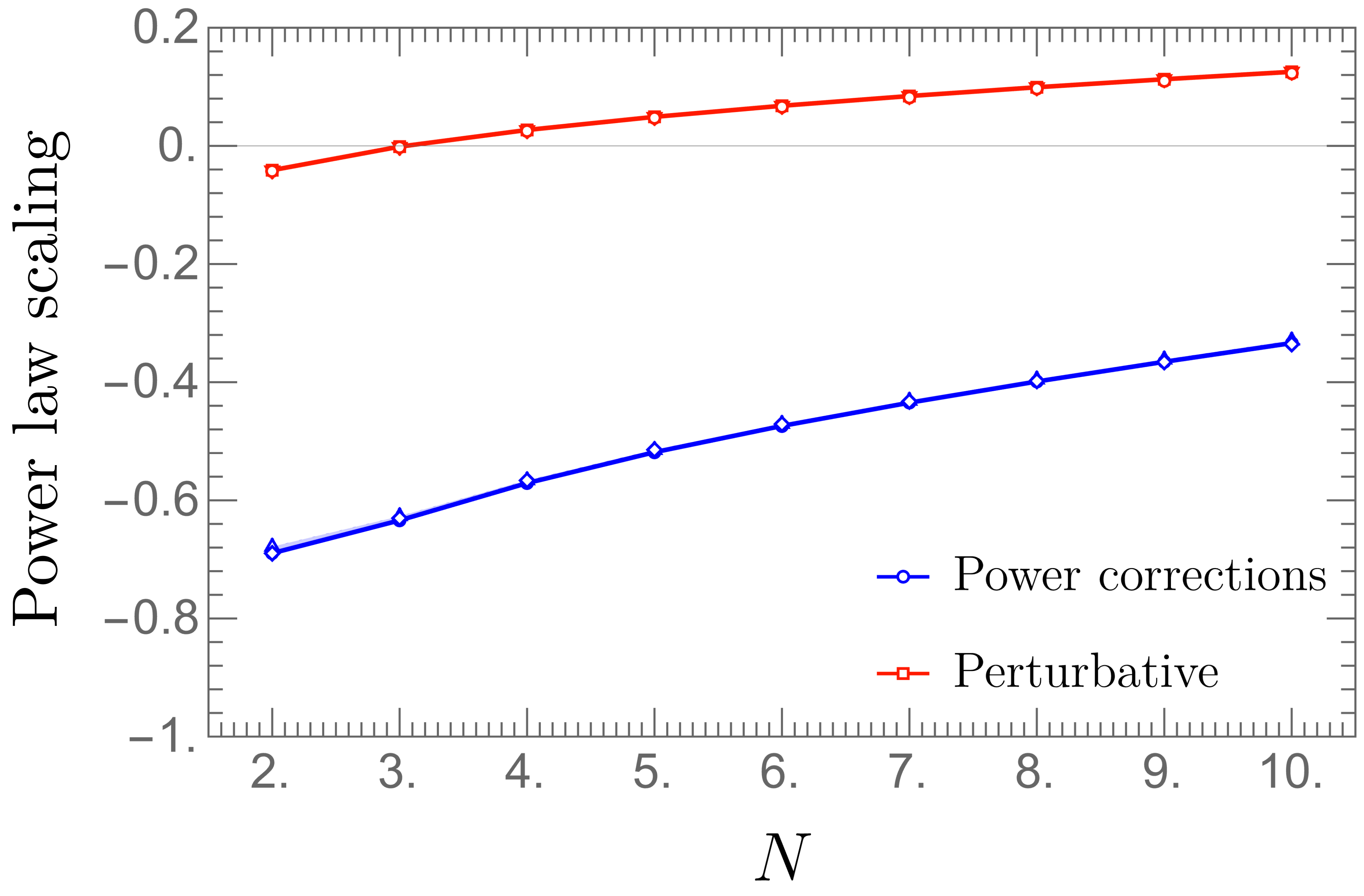




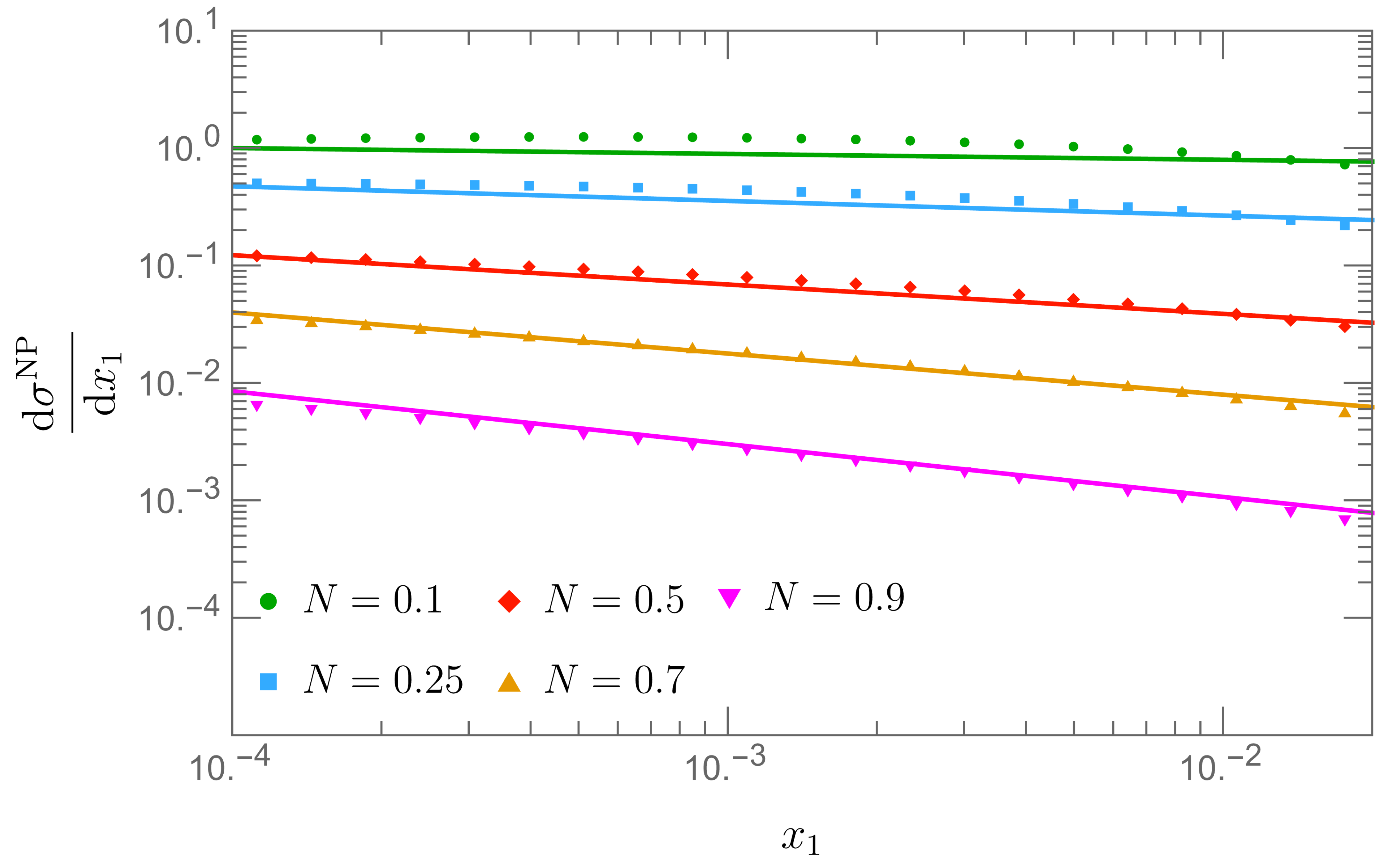
- Normalization is correct, but power-law in x_1 also changes.
- Power-law of nonperturbative: similar to perturbative, qualitatively follows

$$\gamma^{[N]}_{\rm nonp} = -\frac{1}{2} + \gamma^{[N-1]}_{\rm pert}$$

[Lee, Pathak, Stewart, Sun]

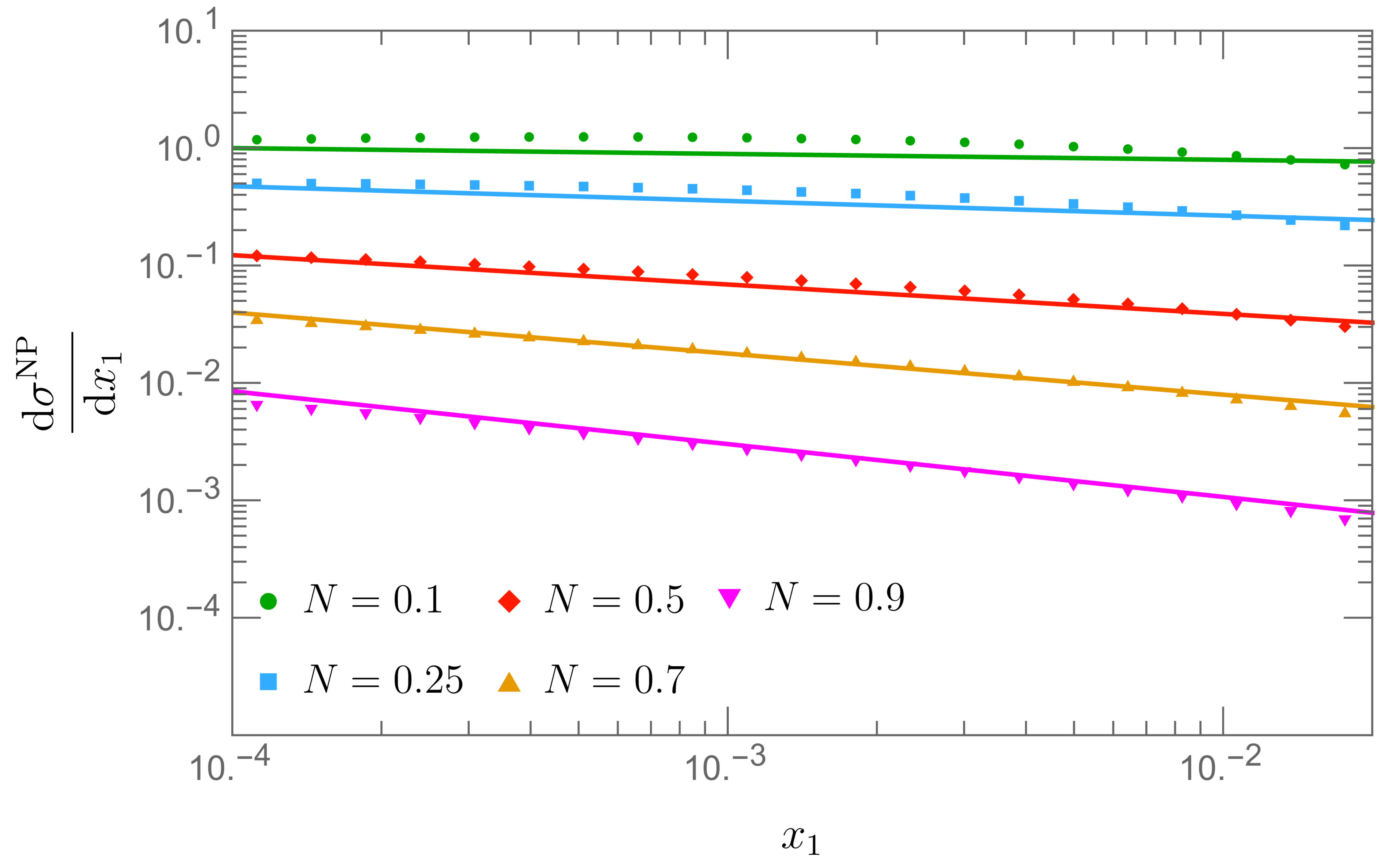
classical scaling changed from $1/x_1$ to $1/x_1^{3/2}$

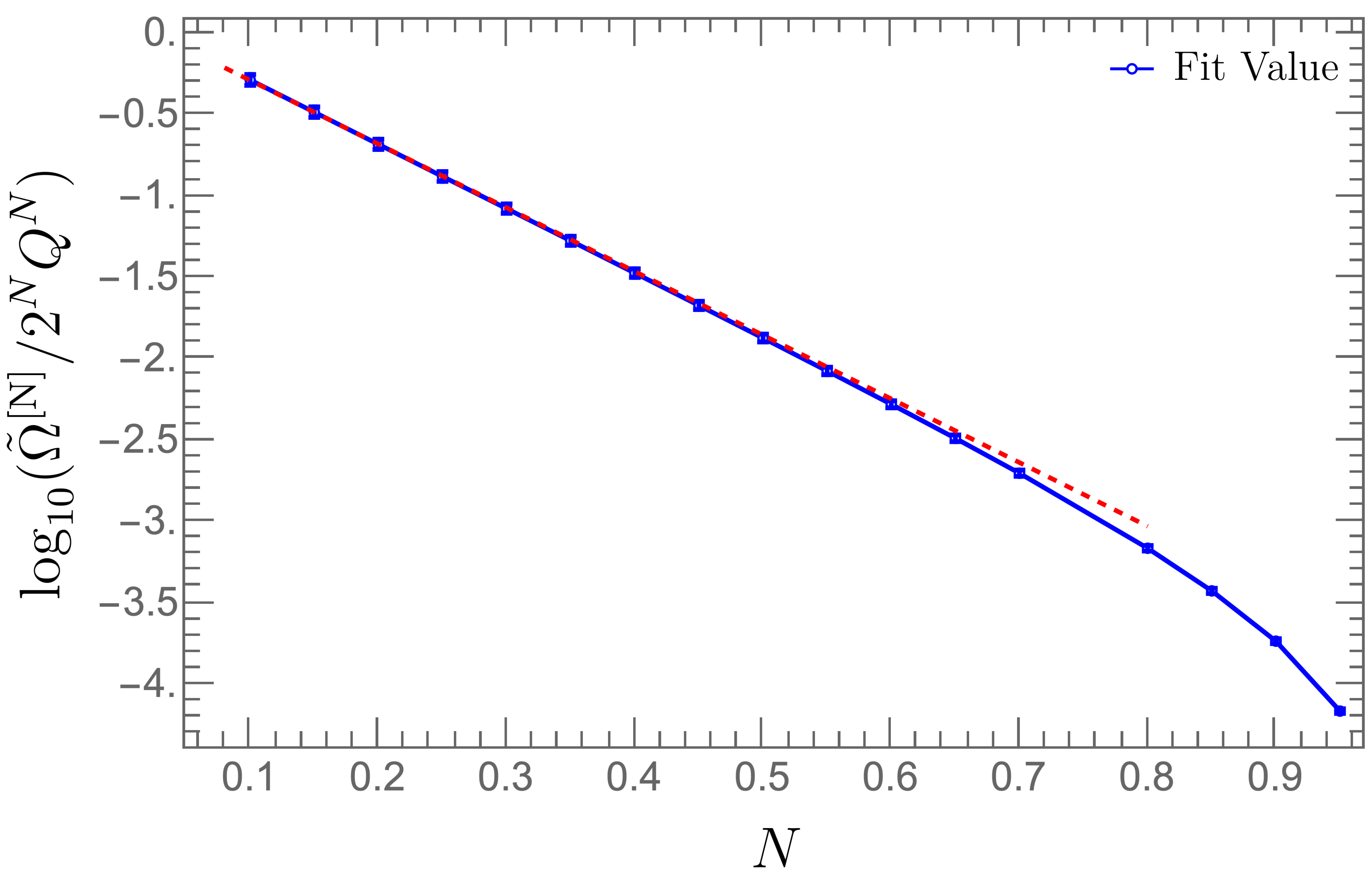
Dependence on N for $N < 1$



- For $N < 1$, modification of classical scaling is observed: $1/x_1^{3/2} \to 1/x_1^{1+N/2}$.

Dependence on N for $N < 1$





- For $N < 1$, modification of classical scaling is observed: $1/x_1^{3/2} \to 1/x_1^{1+N/2}$.
- We fit the nonperturbative parameter, finding $\Omega^{[N]} \approx (\Omega_1)^N$.

Conclusions

- Higher-point correlators are
  - Interesting: α_s, small x in jets, …
  - Feasible: speed up from new parametrization.
- Factorization in back-to-back regime: one modified jet function.
- Nonperturbative effects in collinear regime: new contribution for $N < 1$.

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감사합니다 (*gam-sa-ham-ni-da*)

Backup

Factorization in the collinear regime

- Factorization for standard projected correlator:

$$\int^{R_L} \mathrm{d}R_L' \, \frac{\mathrm{d}\sigma^{[N]}}{\mathrm{d}R_L'} = \int_0^1 \mathrm{d}x \, x^N \, \underbrace{\vec{H}\left(x, \frac{Q}{\mu}\right)}_{\text{hard scattering}} \cdot \underbrace{\vec{J}^{[N]}\left(\ln \frac{R_L x Q}{\mu}\right)}_{\text{jet formation}}$$

[Dixon, Moult, Zhu; Chen, Moult, Zhang, Zhu]

- New parametrization only affects jet function:
  - Difference arises for 3 or more particles in jet, so NNLL effect.
  - Triangle inequality: $R_1 \leq R_L \leq 2R_1$, NNLL implies $R_L = R_1\,[1 + \mathcal{O}(\alpha_s)]$.