# Multi-Jet Final States with Recoil-Free Axes: Generalized $q_T$-Slicing and Dijet Resummation

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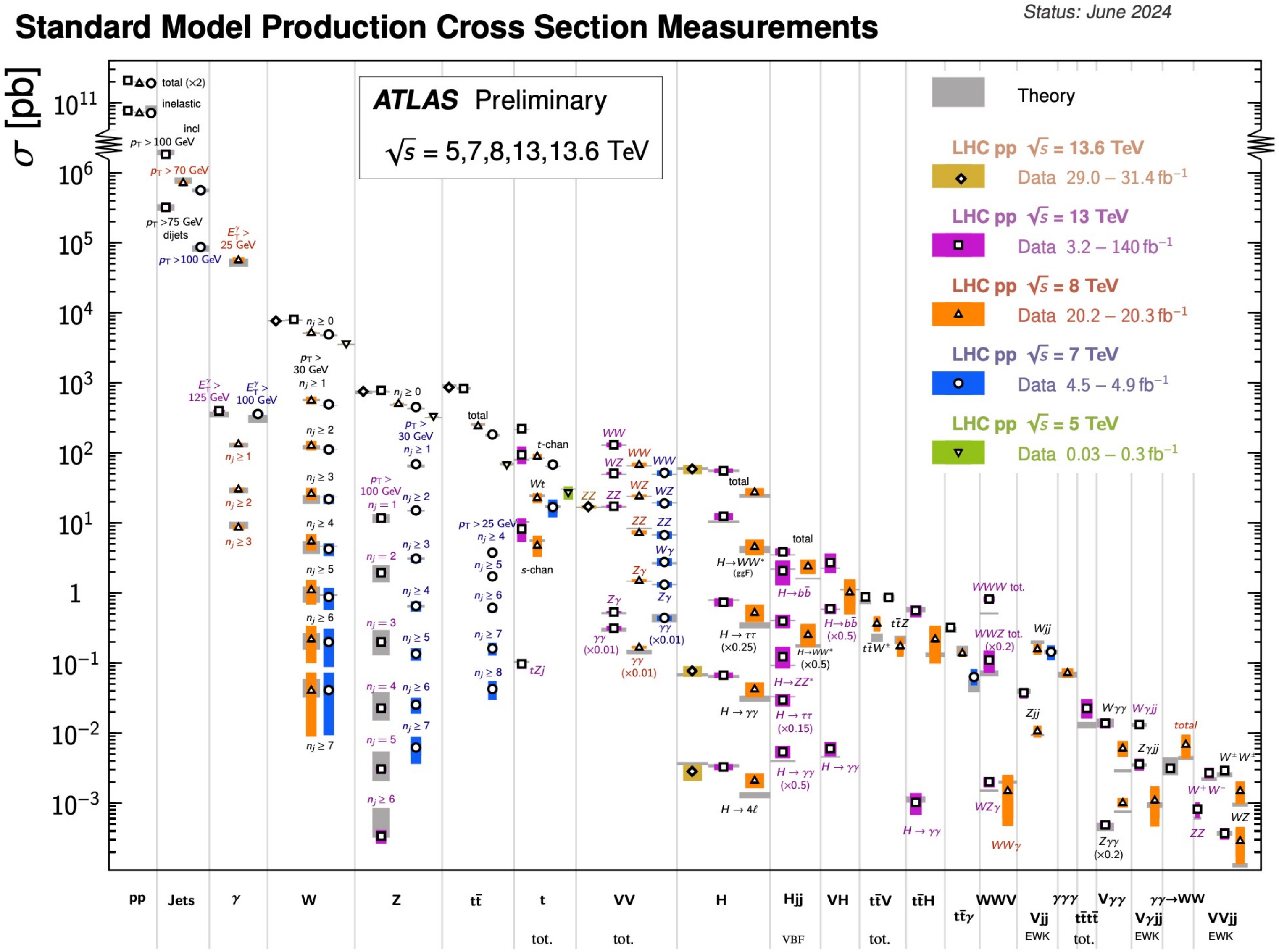
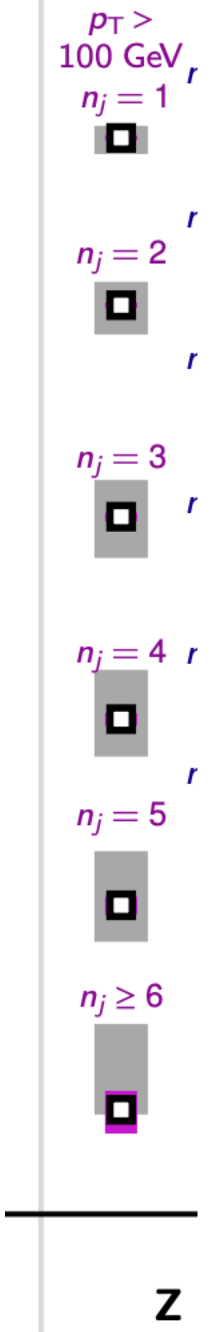
SCET2026, KIAS, Seoul

# Standard Model Summary Plots June 2024

[ATLAS Collaboration ’24]



# Generalized $q_T$-Slicing for Multi-jet Final states

# Slicing Approach

[Fabricius, Schmitt, Kramer, Schierholz '81; Catani, Grazzini '07; Gao, Li, Zhu '13 ...]

$$\frac{\mathrm{d}\sigma^{(m)}_{\mathrm{N^kLO}}}{\mathrm{d}X} = \int_0^\delta \mathrm{d}q_T \frac{\mathrm{d}\sigma^{(m)}_{\mathrm{N^kLO}}}{\mathrm{d}X\mathrm{d}q_T} + \int_\delta^\infty \mathrm{d}q_T \frac{\mathrm{d}\sigma^{(m+1)}_{\mathrm{N^{k-1}LO}}}{\mathrm{d}X\mathrm{d}q_T}$$

- $\delta$ (or $\mathrm{q_T^{cut}}$) plays a role as slicing variable here;
- The factorization formula, obtained e.g. through Soft-Collinear Effective Theory (SCET) [Bauer, Fleming, Luke '00; Bauer, Fleming, Pirjol, Stewart '01], can be used to handle the cancellation of infrared (IR) divergences in

$$\frac{\mathrm{d}\sigma^{(m)}_{\mathrm{N^kLO}}}{\mathrm{d}q_T} \equiv \frac{\mathrm{d}\sigma_{\mathrm{SCET}}}{\mathrm{d}q_T}[1+\mathcal{O}(\delta^p)],$$

- whereas $\frac{\mathrm{d}\sigma^{(m+1)}_{\mathrm{N^{k-1}LO}}}{\mathrm{d}q_T} \equiv \frac{\mathrm{d}\sigma_{\mathrm{QCD}}}{\mathrm{d}q_T}$ is numerically easier to compute.
- Large cancellations between singular and regular terms.

# Modern slicing variables

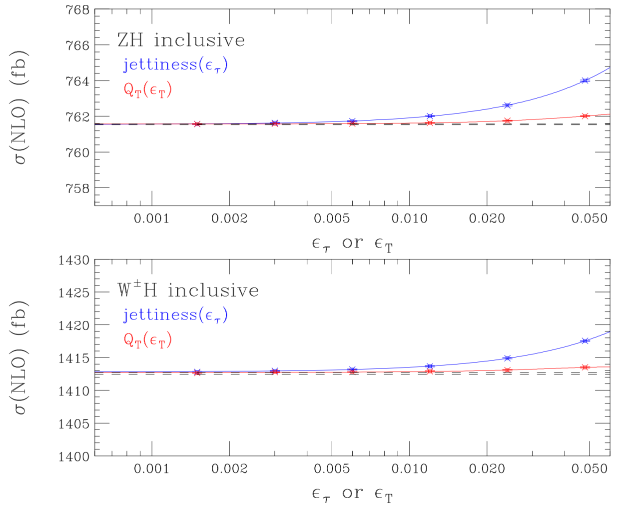
- Transverse momentum of a colorless or colored (but massive) system. [Catani, Grazzini ’07; Catani, Devoto, Grazzini, Kallweit, Mazzitelli, Sargsyan ’19]

For processes with jet production, $q_T$ defined using standard jet axis is unsuitable, because only out-of-jet radiation contributes to SJA $q_T$.

- N-jettiness [Boughezal, Focke, Liu, Petriello ’15; Gaunt, Stahlhofen, Tackmann, Walsh ’15]

Kinematic dependence? [Bell, Dehnadi, Mohrmann, Rahn ’23, Agarwal, Melnikov, Pedron ’24]

Power corrections? [Moult, Rothen, Stewart, Tackmann, Zhu ’17; Ebert, Moult, Stewart, Tackmann, Zhu ’18; Eberta, Tackmann ’20; Campbell, Ellis, Seth ’22; …]




- $k_T$-ness [Buonocore, Grazzini, Haag, Rottoli, Savoini ’22]

- WTA-$q_T$ [RJF, Rahn, Shao, Waalewijn, Wu ’24]

# $q_T$-slicing for Jets

- We propose two generalizations of $q_T$ that can be used for jet processes.

The key ingredient is the use of a recoil-free jet axis!

Utilizing the Winner-Take-All (WTA) scheme to define slicing variables.

- **Standard E-scheme** simply sums the four-momenta of all particles within the jet.
- **Winner-take-all scheme**: [Bertolini, Chan, Thaler '13]

  Clustering step: for the closest emission pair $i$ and $j$:

$$p_{T,r} = p_{T,i} + p_{T,j}, \quad \hat{n}_r = \begin{cases} \hat{n}_i, & p_{T,i} \geq p_{T,j} \\ \hat{n}_j, & p_{T,i} < p_{T,j} \end{cases}$$

In the WTA scheme, radiation inside the jet can also influence the jet axis through momentum conservation, similar to radiation outside the jet, leading to a non-zero $q_T$.

# WTA Transverse Momentum

- Consider a multijet scattering process at LHC: $p(P_a)+p(P_b)\to\sum_{i=1}^{N}J(p_i)+V(p_V)+X.$
- The total transverse momentum of the final states is defined as:

$$\vec{q}_T\equiv\sum_{i=1}^{N}\vec{p}_{i,T}^{\,\mathrm{WTA}}+p_{V,T}=\sum_{i=1}^{N}\Bigg(\sum_{k\in\text{jet-}i}p_{k,T}\Bigg)\vec{n}_{W_i,T}+p_{V,T}.$$

$$\vec{q}_T=\sum_{i=1}^{N}\sum_{k\in\text{jet-}i}\left(p_{k,T}\,\vec{n}_{W_i,T}-\vec{p}_{k,T}\right)+\vec{p}_{V,T}-\vec{p}_{V,T}-\vec{p}_{a,T}-\vec{p}_{b,T}-\vec{p}_{s,T}$$

$$=-\sum_{i=1}^{N}\sum_{k\in\text{jet-}i}\vec{p}_{k,\perp}^{\,(i)}-\vec{p}_{a,T}-\vec{p}_{b,T}-\vec{p}_{s,T},$$

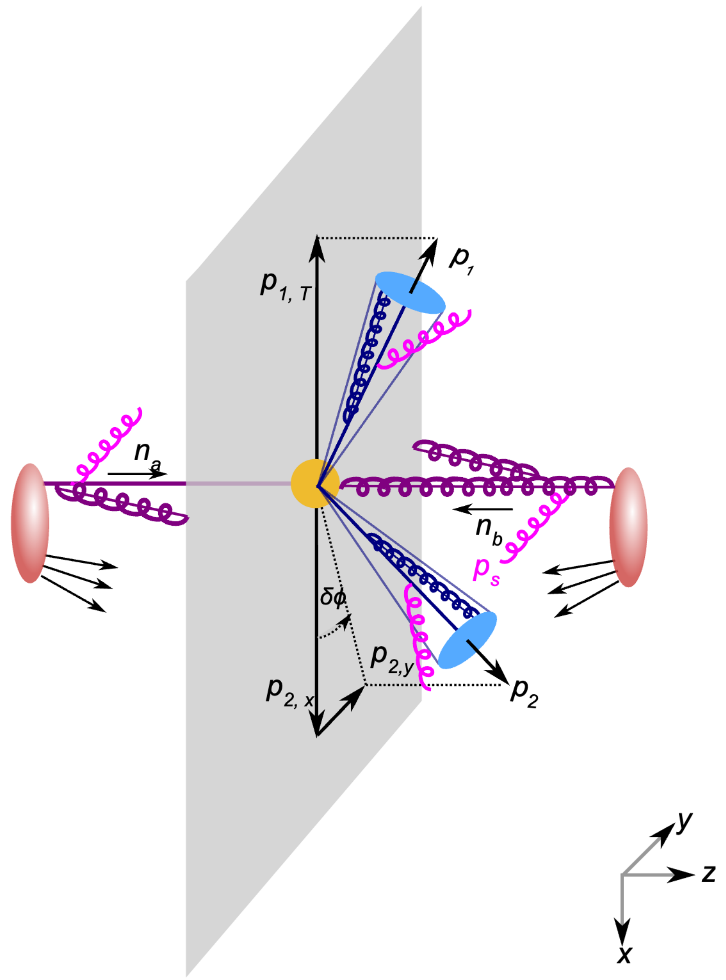
with $\vec{p}_{k,\perp}^{\,(i)}\equiv\vec{p}_{k,T}\;-\;p_{k,T}\;\vec{n}_{W_i,T},\quad k\in\text{jet-}i.$

$$\vec{p}_{k,\perp}^{\,(i)}\cdot\vec{n}_{W_i,T}=0+\mathcal{O}\Big(p_{k,T}\,(\phi_k-\phi_i^{\mathrm{WTA}})^2\Big)$$

→ In-cone contribution $\vec{p}_{k,\perp}^{\,(i)}$ is perpendicular to the jet and the beam axis at leading power.

# Azimuthal decorrelation $\delta\phi \equiv |\pi - |\phi_1^{\mathrm{WTA}} - \phi_2^{\mathrm{WTA}}||.$

- **Planar case:** (e.g., $pp \to 2$ jets)

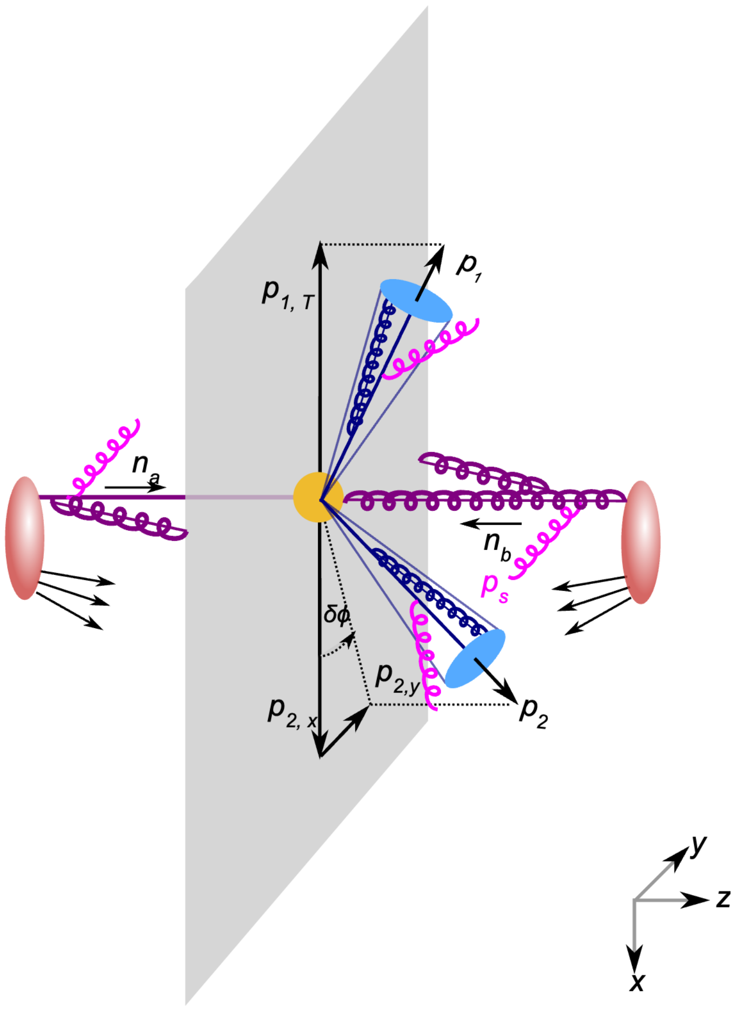



- In the back-to-back limit $(\delta\phi \ll 1)$ $\quad \delta\phi \approx \frac{q_y}{p_T}$
  - **hard**: $p_h^\mu \sim p_T(1,1,1)$,
  - $n_{a,b}$**-collinear**: $p_c^\mu \sim p_T\,(\delta\phi^2, 1, \delta\phi)_{n_i\bar{n}_i}$,
  - **soft**: $p_s^\mu \sim p_T\,(\delta\phi, \delta\phi, \delta\phi)$,
  - $n_{1,2}$**-collinear**: $p_c^\mu \sim p_T\,(\delta\phi^2, 1, \delta\phi)_{n_i\bar{n}_i}$.

- Factorization formula:

$$\frac{\mathrm{d}^4\sigma}{\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}p_T\mathrm{d}q_y} = \sum_{ijk\ell} \frac{x_a x_b}{16\pi\hat{s}^2}\frac{2p_T}{1+\delta_{k\ell}} \mathcal{H}_{ij\to k\ell,JI}(p_T, \eta_1-\eta_2, \mu)$$
$$\times \int_{-\infty}^{\infty}\frac{\mathrm{d}b_y}{2\pi} e^{ib_y q_y} \boldsymbol{S}_{ijk\ell,IJ}(b_y, \eta_1, \eta_2, \mu, \nu)\,\mathcal{J}_k\,(b_y, \omega_1, \mu, \nu)\,\mathcal{J}_\ell\,(b_y, \omega_2, \mu, \nu)$$
$$\times B_{i/p}\,(x_a, b_y, \omega_a, \mu, \nu)\, B_{j/p}\,(x_b, b_y, \omega_b, \mu, \nu)$$

# Azimuthal decorrelation




- Factorization formula ($pp \to 2$jets):

$$\frac{\mathrm{d}^4\sigma}{\mathrm{d}\eta_1 \mathrm{d}\eta_2 \mathrm{d}p_T \mathrm{d}q_y} = \sum_{ijk\ell} \frac{x_a x_b}{16\pi \hat{s}^2} \frac{2p_T}{1+\delta_{k\ell}} \mathcal{H}_{ij\to k\ell,JI}(p_T, \eta_1 - \eta_2, \mu)$$
$$\times \int_{-\infty}^{\infty} \frac{\mathrm{d}b_y}{2\pi} e^{ib_y q_y} \boldsymbol{S}_{ijk\ell,IJ}(b_y, \eta_1, \eta_2, \mu, \nu) \mathcal{J}_k(b_y, \omega_1, \mu, \nu) \mathcal{J}_\ell(b_y, \omega_2, \mu, \nu)$$
$$\times B_{i/p}(x_a, b_y, \omega_a, \mu, \nu) B_{j/p}(x_b, b_y, \omega_b, \mu, \nu)$$

- Factorization ingredients:
  - Standard TMD PDFs $B_{i,j}$: known at N$^3$LO; [Luo, Yang, Zhu, Zhu'19; Ebert, Mistlberger, Vita'20]
  - Soft function $S_{ijk\ell}$ : can be obtained directly from the standard TMD soft function at NNLO; [Gao, Li, Moult, Zhu '19]
  - WTA jet functions $\mathcal{J}_{k,\ell}$: partially known at NNLO. [Gutierrez-Reyes, Scimemi, Waalewijn, Zoppi '19; Bell, Brune, Das, Wald '23; Fang, Gao, Li, Shao '24; Buonocore, Grazzini, Guadagni, Haag, Rottoli '25]

# Transverse momentum imbalance

- We can still use the magnitude of the total transverse momentum $q_T = |\vec{\mathrm{q}}_\mathrm{T}|$ of the color-singlets and jets as a slicing variable, when utilizing the WTA scheme.

$$\vec{q}_T \equiv \sum_{i=1}^{N} \vec{p}_{i,T}^{\,\mathrm{WTA}} + p_{V,T} = -\sum_{i=1}^{N} \sum_{k\in \text{jet-}i} \vec{p}_{k,\perp}^{\,(i)} - \vec{p}_{a,T} - \vec{p}_{b,T} - \vec{p}_{s,T}$$

- $q_T$-slicing can be applied to processes with non-planar kinematics, such as $pp \to Z + 2$jets, $pp \to 3$jets, etc.

- Factorization formula ($pp \to 2$jets):

$$\frac{\mathrm{d}^4\sigma}{\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}p_T\,\mathrm{d}q_T} = \sum_{ijk\ell} \frac{x_a x_b}{16\pi\hat{s}^2} \frac{2p_T}{1+\delta_{k\ell}} \boldsymbol{\mathcal{H}}_{ij\to k\ell,JI}(p_T,\eta_1-\eta_2,\mu)\; q_T \int_0^{2\pi} \mathrm{d}\phi_q \int \frac{\mathrm{d}^2\vec{b}_T}{(2\pi)^2} e^{i\vec{q}_T\cdot\vec{b}_T}$$
$$\times\, \boldsymbol{S}_{ijk\ell,IJ}(\vec{b}_T,\eta_1,\eta_2,R,\mu,\nu)\,\mathcal{J}_k\left(b_y,\omega_1,\mu,\nu\right)\,\mathcal{J}_\ell\left(b_y,\omega_2,\mu,\nu\right)$$
$$\times\, B_{i/p}(x_a,\omega_a,b_T,\mu,\nu)\,B_{j/p}(x_b,\omega_b,b_T,\mu,\nu).$$

- Only soft function is new: Outside the jet ( $\vec{\mathrm{b}}_\mathrm{T}$), inside the jet ($b_x$).

# $q_T$-soft function ($q_T/p_T \ll R \ll 1$)

- The $q_T$-soft function refactorizes into global and "collinear-soft" contributions in small-$R$ limit, while the remaining power corrections appear as a series in $R^{2n}$,

$$\boldsymbol{\mathcal{S}}_{ijk\ell}(\vec{b}_T, R, \eta_1, \eta_2, \mu, \nu) = \boldsymbol{\mathcal{S}}^{\text{global}}_{ijk\ell}(\vec{b}_T, \eta_1, \eta_2, \mu, \nu) \boxed{S_k(\vec{b}_T, \eta_1, R, \mu, \nu)} S_\ell(\vec{b}_T, \eta_2, R, \mu, \nu) + \mathcal{O}(R^{2n}).$$

Encoding collinear-soft and ultra-collinear-soft modes

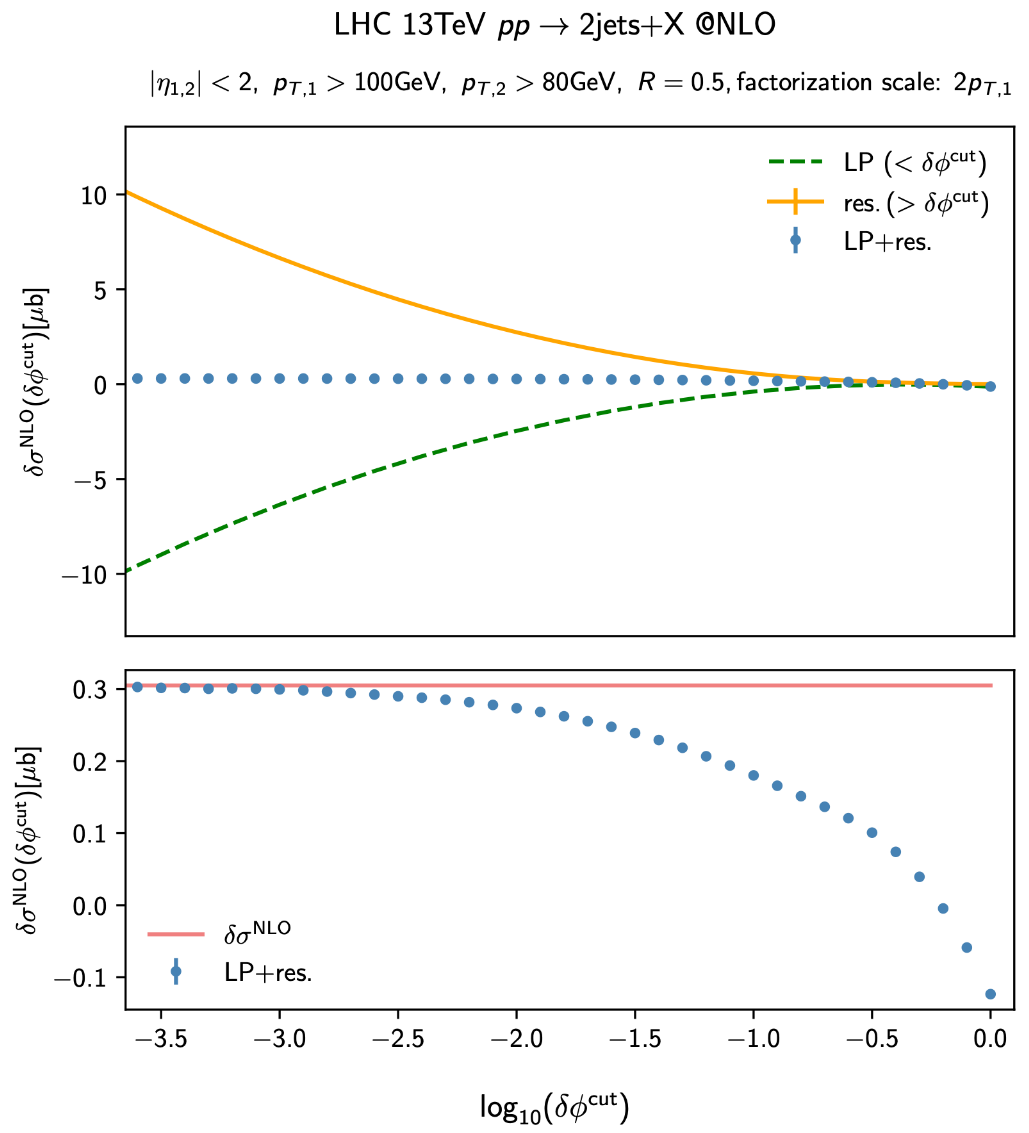
- NLO:

$$\begin{aligned}\hat{\boldsymbol{\mathcal{S}}}^{(1)}_{\text{finite}}(q_T^{\text{cut}}, \eta_1, \eta_2, R, \mu, \nu) = \frac{\alpha_s(\mu)}{4\pi} \Bigg\{ &-4L_\mu^2 \sum_i \mathbf{T}_i^2 + L_\mu \bigg[ 4 \ln \frac{\mu^2}{\nu^2} \sum_i \mathbf{T}_i^2 + 8 \sum_{i<j} \mathbf{T}_i \cdot \mathbf{T}_j \ln \frac{n_i \cdot n_j}{2} \\ &- 8 \ln 2 \, (\mathbf{T}_a + \mathbf{T}_b) \cdot (\mathbf{T}_1 + \mathbf{T}_2) - 16 \ln 2 \, \mathbf{T}_1 \cdot \mathbf{T}_2 \bigg] - \frac{\pi^2}{6} \sum_i \mathbf{T}_i^2 \\ &+ [(\mathbf{T}_a + \mathbf{T}_b) \cdot (\mathbf{T}_1 + \mathbf{T}_2) + 2 \, \mathbf{T}_1 \cdot \mathbf{T}_2] \left( 4 \ln 2 \ln \frac{\mu^2}{\nu^2} + \frac{\pi^2}{3} + 4 \ln^2 \frac{R}{2} \right) \\ &+ \sum_{j \in \text{jets}} (\mathbf{T}_a + \mathbf{T}_b) \cdot \mathbf{T}_j \; 8 \ln 2 \ln (2 \cosh \eta_j) \\ &+ \mathbf{T}_1 \cdot \mathbf{T}_2 \left[ 8 \ln 2 \ln(4 \cosh \eta_1 \cosh \eta_2) - 2 \ln^2(2 + 2 \cosh(\eta_1 - \eta_2)) + 2(\eta_1 - \eta_2)^2 \right] \\ &- \sum \mathbf{T}_i \cdot \mathbf{T}_j \, S^{\text{corr}}_{ij}(\eta_1, \eta_2, R) \Bigg\}\end{aligned}$$

$R$-correction for jet-jet dipole:

$$\begin{aligned}S^{\text{corr}}_{12}(\eta_1, \eta_2, R) = &-2R^2 \ln \frac{R}{2} \tanh^2 \left( \frac{\eta_1 - \eta_2}{2} \right) + R^2 \left[ \frac{7}{3} - \frac{6}{1 + \cosh(\eta_1 - \eta_2)} \right] \\ &+ R^4 \left[ \frac{49}{720} - \frac{e^{\eta_1 + \eta_2} \left( 3e^{2\eta_1} + 3e^{2\eta_2} - 8e^{\eta_1 + \eta_2} \right)}{2(e^{\eta_1} + e^{\eta_2})^4} - \ln \left( \frac{R}{2} \right) \frac{\left( e^{2\eta_1} + e^{2\eta_2} - 10e^{\eta_1 + \eta_2} \right)^2}{36(e^{\eta_1} + e^{\eta_2})^4} \right] + \mathcal{O}(R^6)\end{aligned}$$

# $q_y$-slicing for $pp \to$ 2jets

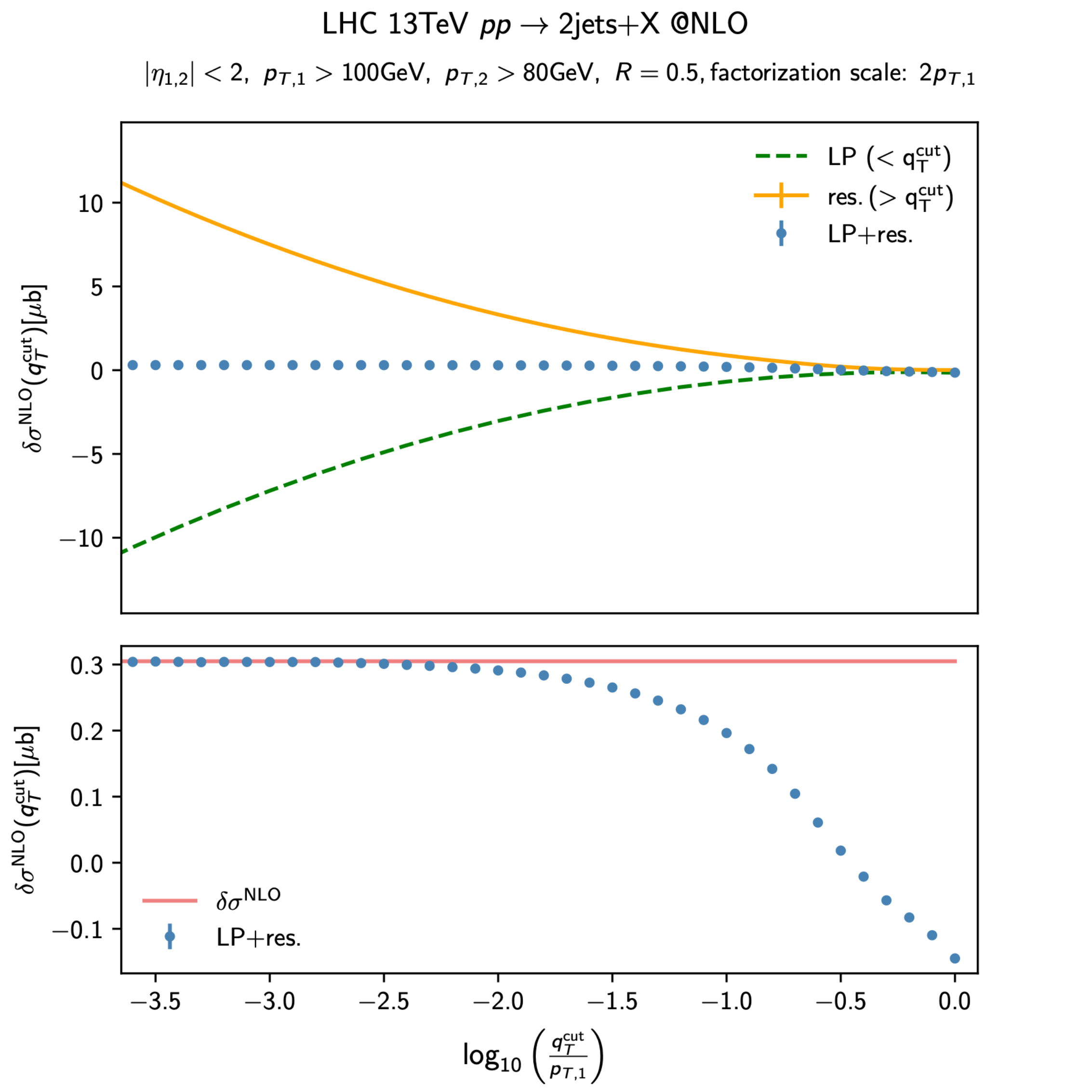



- Slicing capitalizes on this,

$$\frac{d\sigma}{dX} = \int_0^{\delta} dq_y \frac{d\sigma_{LP}}{dX\, dq_y}[1 + \mathcal{O}(\delta^p)] + \int_{\delta}^{\infty} dq_y \frac{d\sigma_{res.}}{dX\, dq_y}.$$

- $\sigma_{res.}$ is obtained from NLOJET++ [Nagy, Trocsanyi '01] and FASTJET [Cacciari, Salam, Soyez '11].
- Jets are defined in a standard way with partons clustered by anti-$k_T$ algorithm and momentum recombined by standard $E$-scheme.
- Large cancellation and nice convergence at NLO.

# $q_T$-slicing for $pp \to 2$jets

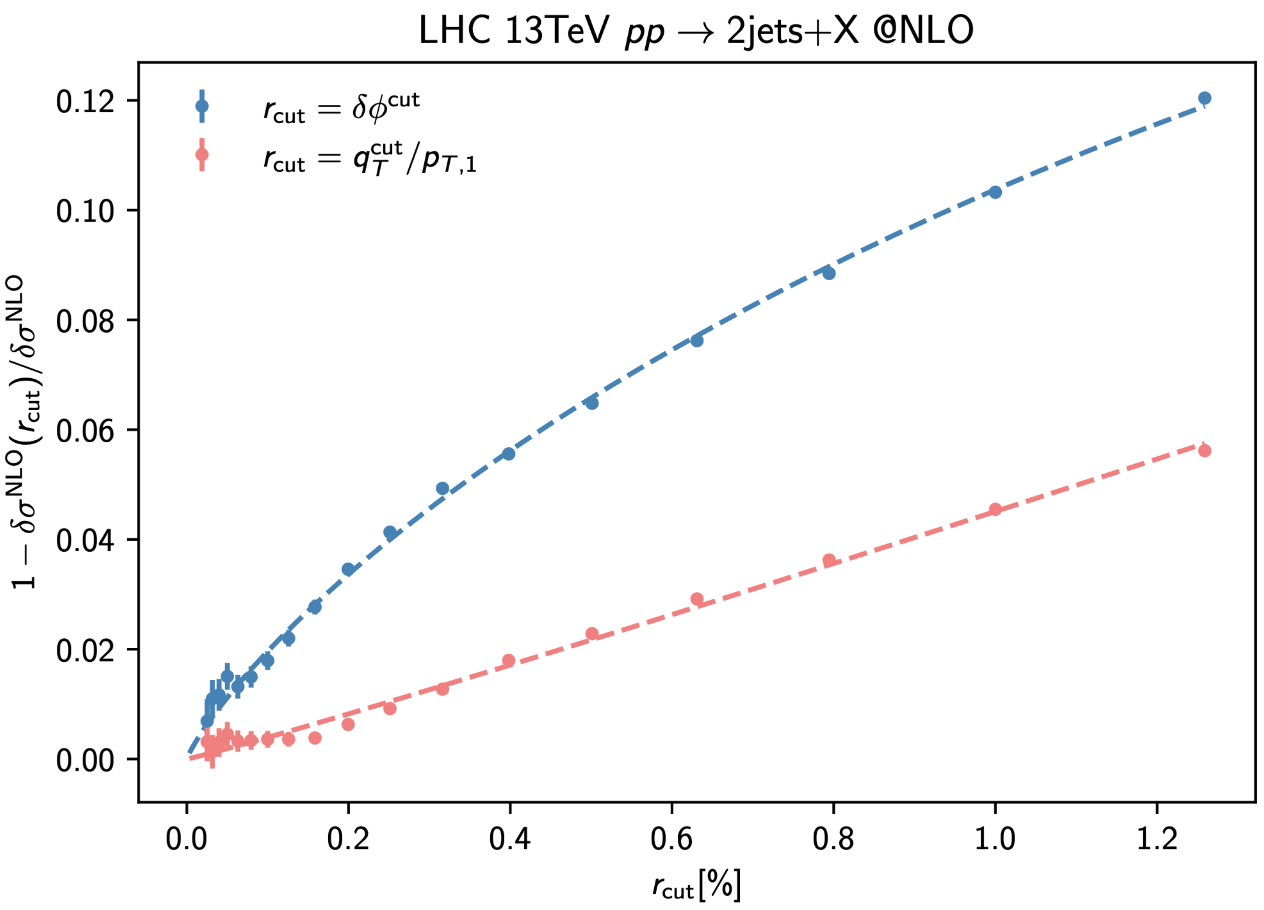



- Slicing capitalizes on this,

$$\frac{d\sigma}{dX} = \int_0^{\delta} dq_T \frac{d\sigma_{LP}}{dX\,dq_T}[1 + \mathcal{O}(\delta^p)] + \int_{\delta}^{\infty} dq_T \frac{d\sigma_{res.}}{dX\,dq_T}.$$

- We include finite terms up to $\mathcal{O}(R^4)$, achieving a 1% precision for the cross section at $R = 0.5$.

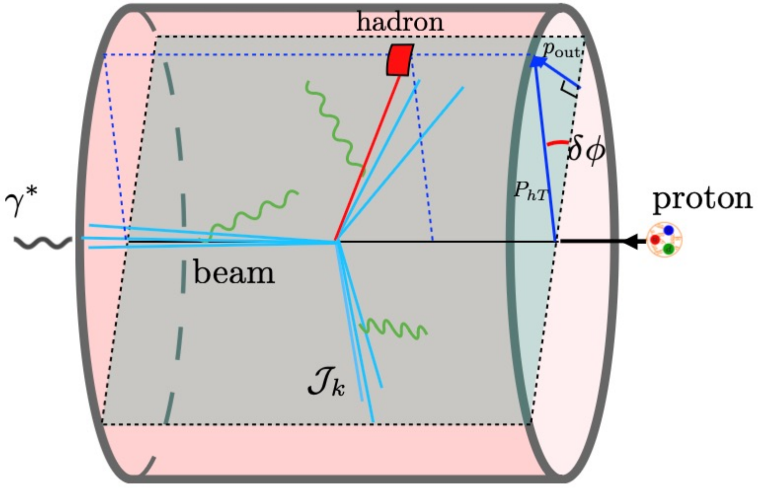
# Comparison between two slicing variables

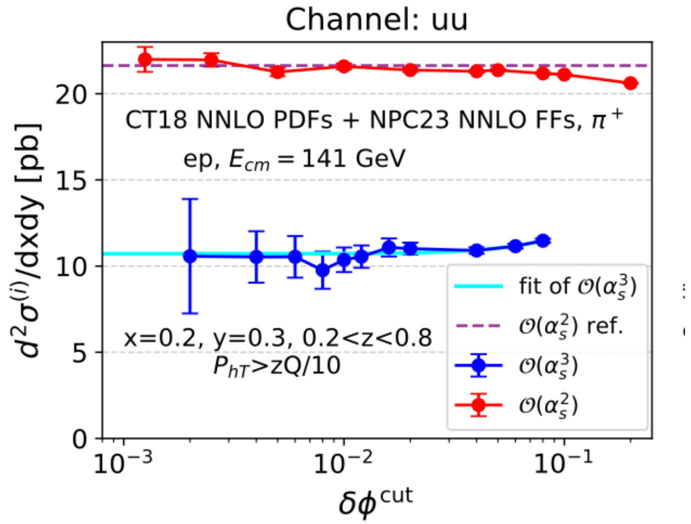



- The curves are obtained from fitting to $a\, r_{\text{cut}} \ln r_{\text{cut}} + b\, r_{\text{cut}}$.
- $q_T$ converges faster than $q_y$ slicing at the expense of a more complicated soft function, and can also be extended to non-planar Born processes.

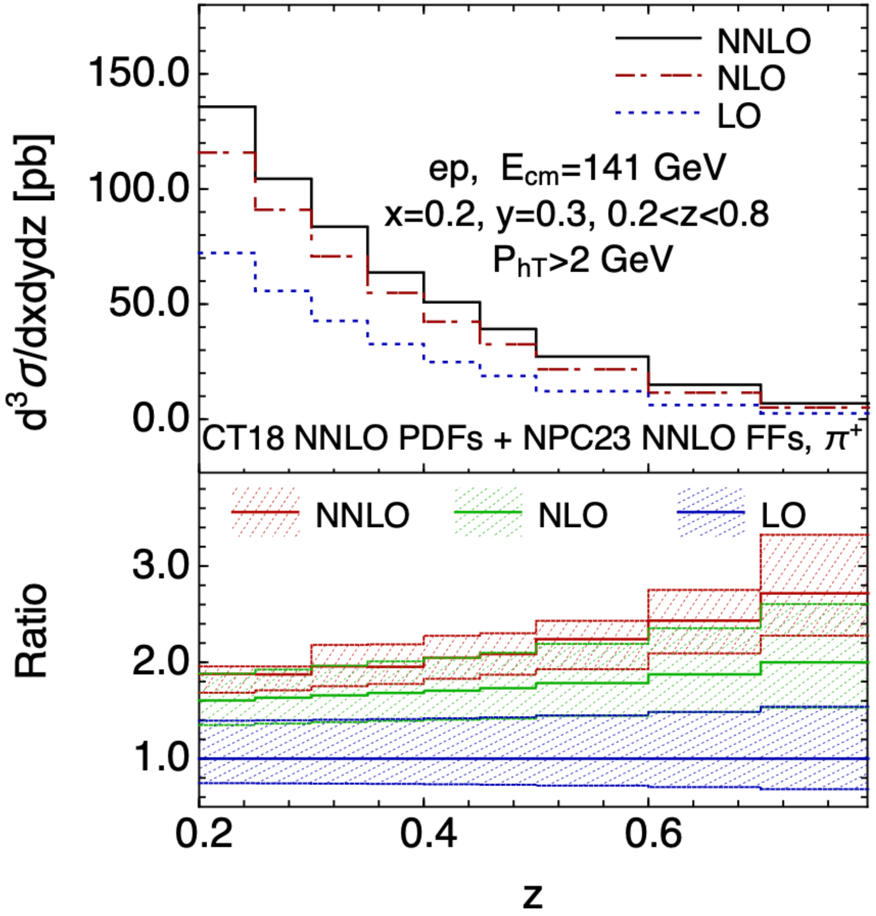
# NNLO WTA $\delta\phi$-slicing in SIDIS

[Dong, Fang, Gao, Li, Shao, Zhu ’26] [2602.22972]




$$e + p \to e + j + h + X$$



# Dijet resummation for WTA azimuthal decorrelation and transverse momentum imbalance

# Non-global logarithms (NGLs)

- For the azimuthal decorrelation $\delta\phi$, the recoil-free WTA recombination scheme eliminates non-global logarithms (NGLs) entirely. [Chien, Rahn, Velzen, Shao, Waalewijn, Wu '20; Chien, Rahn, Shao, Waalewijn, Wu '22]

- For the transverse momentum imbalance $q_T$ , the standard NGLs of $q_T$ disappear, but NGLs of $R$ emerge specifically in the small jet radius limit ($R \ll 1$).

# Resummation formula for $\boldsymbol{\delta\phi}$

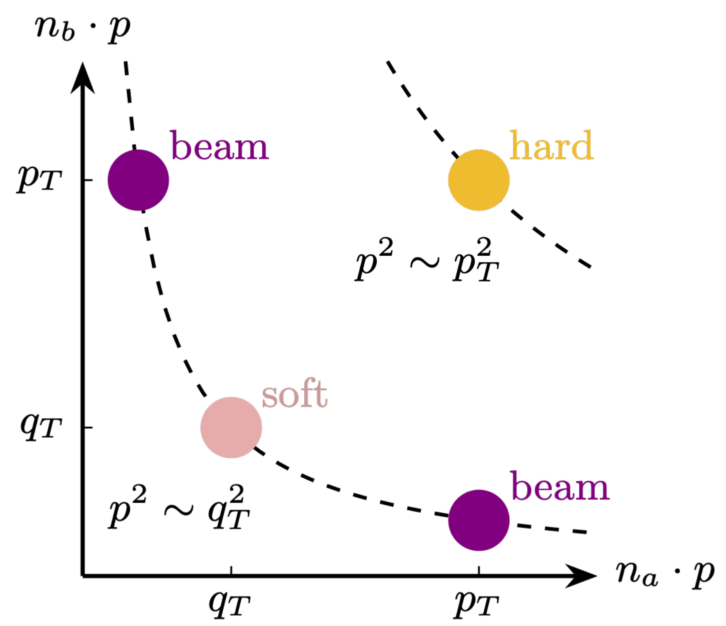
RRG: Jet & beam $\nu_i$ → Soft $\nu_s$

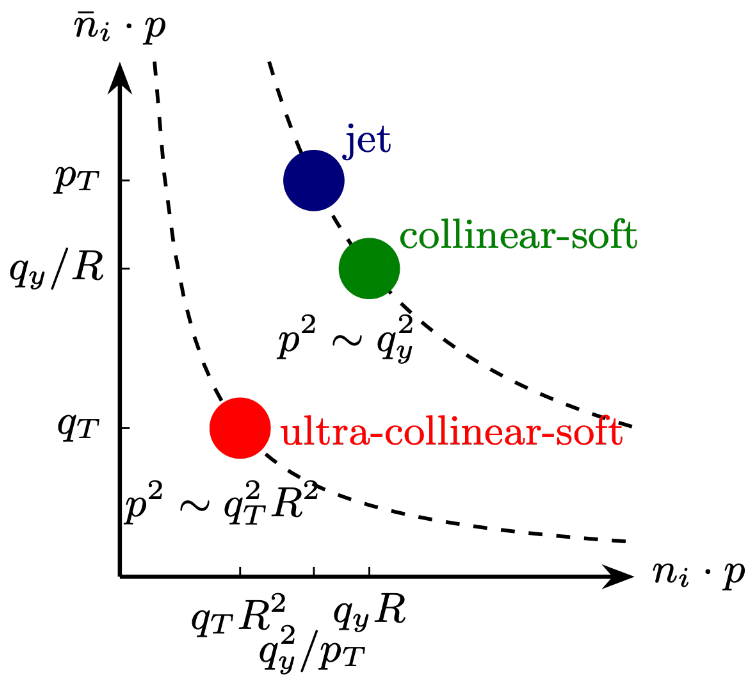
RG: Hard $\mu_h$ → Soft, Jet & beam $\mu_f$

[Chiu, Jain, Neill, Rothstein '11]

$$
\begin{aligned}
\frac{\mathrm{d}^4\sigma}{\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}p_T\,\mathrm{d}q_y} = & \sum_{ijk\ell} \frac{x_a x_b}{16\pi\hat{s}^2}\frac{2p_T}{1+\delta_{k\ell}} \int_0^\infty \frac{\mathrm{d}b_y}{\pi}\cos(b_y q_y) \prod_{i=a,b,1,2} \left(\frac{\nu_s}{\nu_i}\right)^{\Gamma_\nu^i(b_y,\mu_f)} \\
& \times \sum_{KK'} \exp\left\{\int_{\mu_h}^{\mu_f}\frac{\mathrm{d}\mu}{\mu}\left[\gamma_{\text{cusp}}(\alpha_s)\left(C_H \ln\frac{\hat{s}}{\mu^2}+\lambda_K+\lambda^*_{K'}\right)+2\gamma_H(\alpha_s)\right]\right\} \\
& \times \boldsymbol{\mathcal{H}}_{ij\to k\ell,KK'}(p_T,\eta_1-\eta_2,\mu_h)\,\boldsymbol{S}_{ijk\ell,K'K}(b_y,\eta_1,\eta_2,\mu_f,\nu_s) \\
& \times B_{i/p}(x_a,\omega_a,b_y,\mu_f,\nu_a)\,B_{j/p}(x_b,\omega_b,b_y,\mu_f,\nu_b) \\
& \times \mathcal{J}_k(b_y,\omega_1,\mu_f,\nu_1)\,\mathcal{J}_\ell(b_y,\omega_2,\mu_f,\nu_2) \\
& \times \exp\left[-S^i_{\text{NP}}(b_y,Q_0,\omega_a)-S^j_{\text{NP}}(b_y,Q_0,\omega_b)\right].
\end{aligned}
$$

# SCET modes for $q_T$ distribution




$(i = 1 \text{ or } 2)$

- **hard:** $p_h^\mu \sim p_T(1,1,1)$,
- **$n_{a,b}$-collinear:** $p_{c_i}^\mu \sim (q_T^2/p_T, p_T, q_T)_{n_i\bar{n}_i}$,
- **$n_{1,2}$-collinear:** $p_{c_i}^\mu \sim (q_y^2/p_T, p_T, q_y)_{n_i\bar{n}_i}$.
- **soft:** $p_s^\mu \sim q_T\,(1,1,1)$,
- **$n_{1,2}$-collinear-soft:** $p_{cs}^\mu \sim |q_y|/R\,(R^2, 1, R)_{n_i\bar{n}_i}$,
- **$n_{1,2}$-ultra-collinear-soft:** $p_{ucs}^\mu \sim q_T\,(R^2, 1, R)_{n_i\bar{n}_i}$.

# The function $S_i$ $(q_T/p_T \ll R \ll 1)$

- The $q_T$-soft function refactorizes into global and "collinear-soft" contributions in small-$R$ limit, while the remaining power corrections appear as a series in $R^{2n}$,

$$\boldsymbol{S}_{ijk\ell}(\vec{b}_T, R, \eta_1, \eta_2, \mu, \nu) = \boldsymbol{S}^{\text{global}}_{ijk\ell}(\vec{b}_T, \eta_1, \eta_2, \mu, \nu) \boxed{S_k(\vec{b}_T, \eta_1, R, \mu, \nu)} S_\ell(\vec{b}_T, \eta_2, R, \mu, \nu) + \mathcal{O}(R^{2n}).$$

Encoding the collinear-soft and ultra-collinear-soft modes

- **Non-Global Logarithms (NGLs):** $\mathrm{S}_k(\mathrm{S}_\ell)$ contains NGLs of the jet radius.
- Calculation Method 1: These can be computed using the **multiplicity-interacting EFT framework**. [Becher, Neubert, Rothen, Shao '15, '16; Becher, Pecjak, Shao '16; Larkoski, Moult, Neill '15]
- Calculation Method 2: Boosting along the jet axis reduces $\mathrm{S}_k(\mathrm{S}_\ell)$ to the **hemisphere soft function**, separating the dependence:
  - **In-cone:** depends on $b_\perp$
  - **Out-of-cone:** depends on $b_-$

# Refactorization of $S_i$

- For the momentum $k$ of ultra-collinear-soft mode, we have $k^- \gg k^+, k_\perp$.
- Under this hierarchy, the Fourier exponent associated with the out-of-cone measurement admits the power expansion

$$\underbrace{\vec{b}_T \cdot \vec{k}_T}_{\text{beam coordinates}} = -b \cdot k \approx \underbrace{-\frac{1}{2} b^+ k^-}_{\text{jet coordinates}} ,$$

with $b^\mu = (0, b_x, b_y, 0)$ and $b^+ = -b_x n_{i,x}$.

- Momentum decomposition along the jet axis: $p^\mu = (n_i \cdot p)\frac{\bar{n}_i^\mu}{2} + (\bar{n}_i \cdot p)\frac{n_i^\mu}{2} + p_\perp^\mu \equiv (n_i \cdot p,\ \bar{n}_i \cdot p,\ p_\perp^\mu)_{n_i \bar{n}_i}$

- Applying the factorization framework for non-global observables in the small $R$ limit: [Becher, Neubert, Rothen, Shao '15, '16; Becher, Pecjak, Shao '16]

$$S_i(\vec{b}_T, \eta_i, R, \mu, \nu) = \sum_{m_i=0}^{\infty} \prod_{j=1}^{m_i} \int \frac{d\Omega(\vec{u}_j)}{4\pi} \frac{1}{d_i} \mathrm{Tr}\left[\boldsymbol{\mathcal{S}}^{\text{cs}}_{m_i}(\{n_i, \bar{n}_i, \underline{u}\}, b_y, R, \mu, \nu) \boldsymbol{\mathcal{S}}^{\text{ucs}}_{m_i}(\{n_i, \bar{n}_i, \underline{u}\}, b_x, R, \mu)\right]$$

$$:= \sum_{m_i=0}^{\infty} \left\langle \boldsymbol{\mathcal{S}}^{\text{ucs}}_{m_i}(\{n_i, \bar{n}_i, \underline{u}\}, b_x, R, \mu) \otimes \boldsymbol{\mathcal{S}}^{\text{cs}}_{m_i}(\{n_i, \bar{n}_i, \underline{u}\}, b_y, R, \mu, \nu) \right\rangle .$$

# Collinear-soft and ultra-collinear-soft functions

$$\boldsymbol{S}^{\rm cs}_{m_i}(\{n_i, \bar{n}_i, \underline{u}\}, b_y, R, \mu, \nu) =$$

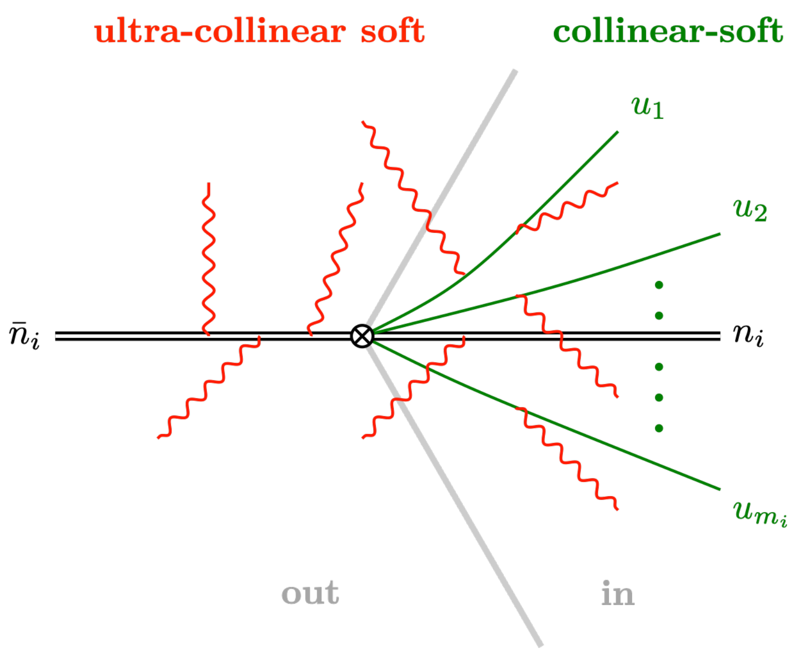
$$\prod_{j=1}^{m_i} \int \frac{\mathrm{d}E_j E_j^{d-3}}{(2\pi)^{d-2}} \left|\mathcal{M}^{cs}_{m_i}(n_i, \bar{n}_i, \{\underline{p}\})\right\rangle \left\langle \mathcal{M}^{cs}_{m_i}(n_i, \bar{n}_i, \{\underline{p}\})\right| \exp\left(i b_y \sum_{k=1}^{m_i} p_{k,y}\right) \Theta^i_{\rm in}(\{\underline{p}\}),$$

Collinear-soft amplitudes: $\left|\mathcal{M}^{cs}_{m_i}(n_i, \bar{n}_i, \{\underline{p}\})\right\rangle = \langle \{\underline{p}\} | S(n_i) S^\dagger(\bar{n}_i) | 0 \rangle$

In-cone constraint: $\Theta^i_{\rm in}(\{\underline{p}\}) = \prod_{j=1}^{m_i} \Theta\left(R_i^2 - \frac{n_i \cdot p_j}{\bar{n}_i \cdot p_j}\right), \quad R_i = \frac{R}{2\cosh \eta_i}.$

$$\boldsymbol{S}^{\rm ucs}_{m_i}(\{n_i, \bar{n}_i, \underline{u}\}, b_x, R, \mu) = \sum_{X_{ucs}} \langle 0 | U^\dagger_{m_i}(u_{m_i}) \cdots U^\dagger_1(u_1) U^\dagger_b(\bar{n}_i) U^\dagger_a(n_i) | X_{ucs} \rangle$$

$$\langle X_{ucs} | U_a(n_i) U_b(\bar{n}_i) U_1(u_1) \cdots U_{m_i}(u_{m_i}) | 0 \rangle \exp\left(-\frac{i}{2} b^+ \sum_{j \notin \text{jet}-i} k_j^-\right)$$



# Combination of the collinear-soft and ultra-collinear-soft functions

- The perturbative expansion of $S_i$ is organized as

$$S_i(b_\perp, b^+) = 1 + \frac{Z_\alpha \alpha_s(\mu)}{4\pi} S_i^{(1)}(b_\perp, b^+) + \left(\frac{Z_\alpha \alpha_s(\mu)}{4\pi}\right)^2 S_i^{(2)}(b_\perp, b^+) + \mathcal{O}(\alpha_s^3)$$

- NLO $S_i$: $$S_i^{(1)}(b_\perp, b^+) = \langle \boldsymbol{S}_0^{\text{ucs}(1)} \boldsymbol{S}_0^{\text{cs}(0)} \rangle + \langle \boldsymbol{S}_0^{\text{ucs}(0)} \boldsymbol{S}_0^{\text{cs}(0)} \rangle + \langle \boldsymbol{S}_1^{\text{ucs}(0)} \otimes \boldsymbol{S}_1^{\text{cs}(1)} \rangle = \langle \boldsymbol{S}_0^{\text{ucs}(1)} \rangle + \langle \boldsymbol{1} \otimes \boldsymbol{S}_1^{\text{cs}(1)} \rangle$$

with: $$\langle \boldsymbol{1} \otimes \boldsymbol{S}_1^{\text{cs}(1)}(b_\perp) \rangle = C_i \left(\frac{\mu |b_\perp|}{b_0}\right)^{2\epsilon} \left(\frac{\nu |b_\perp| R_i}{b_0}\right)^{\eta} h_{\text{in}}, \quad \langle \boldsymbol{S}_0^{\text{ucs}(1)}(b^+) \rangle = C_i \left(\frac{i\, b^+ \mu}{b_0 R_i}\right)^{2\epsilon} s_{\text{out}},$$

$$h_{\text{in}} = \frac{2}{\epsilon^2} - \frac{4}{\eta\epsilon} - \frac{\pi^2}{6} + \left(-\frac{\eta}{\epsilon^3} + \frac{\pi^2\eta}{12\epsilon} - \frac{\zeta_3}{3}\eta - \frac{\pi^2\epsilon}{3\eta} - \frac{4\zeta_3}{3}\epsilon - \frac{17\pi^4}{1440}\eta\epsilon - \frac{4\zeta_3\epsilon^2}{3\eta} - \frac{3\pi^4}{80}\epsilon^2 - \frac{\pi^4\epsilon^3}{40\eta}\right)$$

$$s_{\text{out}} = -\frac{2}{\epsilon^2} - \frac{\pi^2}{2} + \left(-\frac{14\zeta_3}{3}\epsilon - \frac{7\pi^4}{48}\epsilon^2\right)$$

# Combination of the collinear-soft and ultra-collinear-soft functions

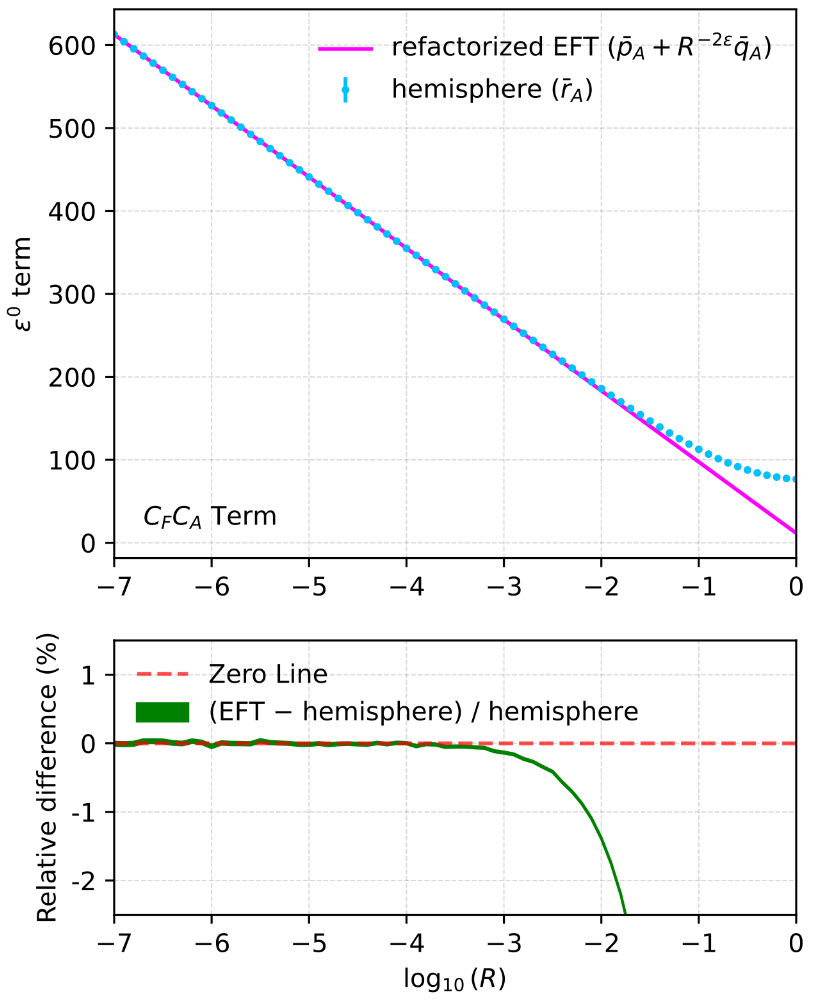
- NNLO $S_i$:

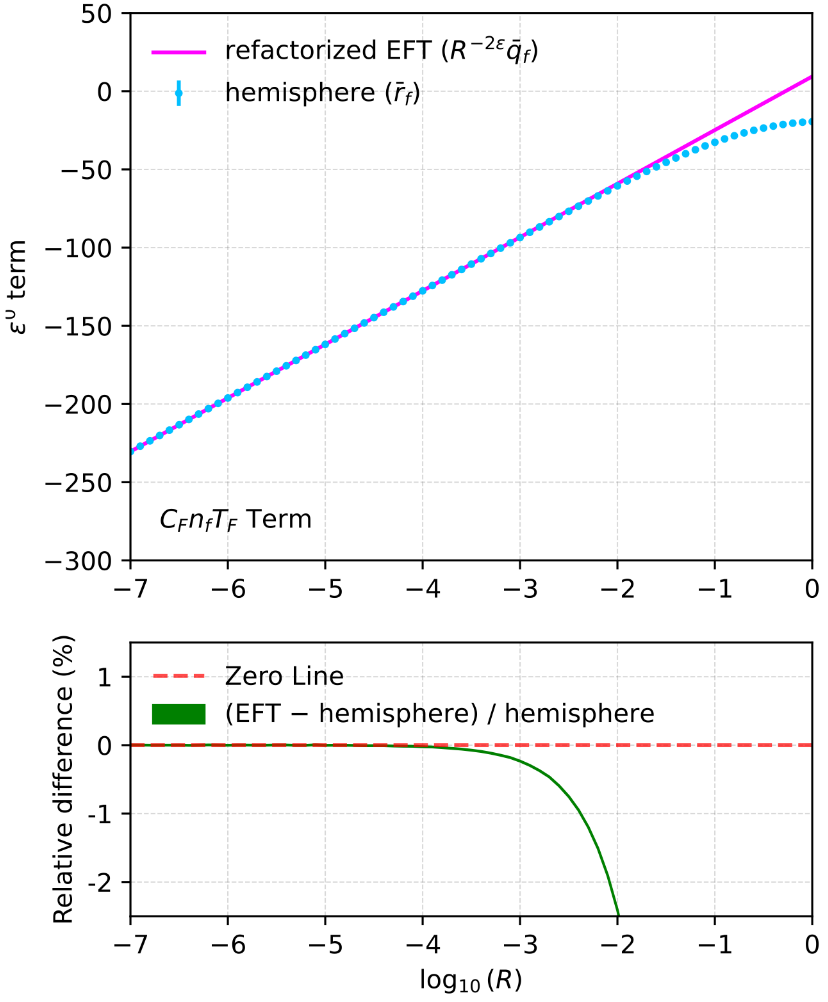
$$
\begin{aligned}
S_i^{(2)}(b_\perp, b^+) = & \left(\frac{\mu|b_\perp|}{b_0}\right)^{4\epsilon}\left(\frac{\nu|b_\perp|R_i}{b_0}\right)^{\eta}\omega^2\left[\omega^2\left(\frac{\nu|b_\perp|R_i}{b_0}\right)^{\eta}C_i^2\frac{h_{\rm in}^2}{2}+C_iC_A(h_A+v_A^{\rm in})+C_in_fT_Fh_f\right] \\
& +\left(\frac{ib^+\mu}{b_0R_i}\right)^{4\epsilon}\left[C_i^2\frac{s_{\rm out}^2}{2}+C_iC_A(g_A+v_A^{\rm out})+C_in_fT_Fg_f\right] \\
& +\left(\frac{ib^+\mu}{b_0R_i}\right)^{4\epsilon}\left[C_iC_A(s_A-g_A-v_A^{\rm out})+C_in_fT_F(s_f-g_f)\right] \\
& +\left(\frac{\mu|b_\perp|}{b_0}\right)^{2\epsilon}\left(\frac{ib^+\mu}{b_0R_i}\right)^{2\epsilon}\left[\left(\frac{\nu|b_\perp|R_i}{b_0}\right)^{\eta}\omega^2C_i^2h_{\rm in}s_{\rm out}+C_iC_Ap_A\right].
\end{aligned}
$$

  - Double-real ultra-collinear-soft emissions, where one emission is in-cone and unobserved, while the other is emitted outside the jet cone and measured:

$$
q_A = s_A - g_A - v_A^{\rm out} = \left(-\frac{2}{3}+\frac{22\pi^2}{9}-4\zeta_3\right)\frac{1}{\epsilon}+\frac{40}{9}-\frac{134\pi^2}{27}+\frac{44\zeta_3}{3}+\frac{8\pi^4}{45}, \quad q_f = s_f - g_f = \left(\frac{4}{3}-\frac{8\pi^2}{9}\right)\frac{1}{\epsilon}-\frac{68}{9}+\frac{64\pi^2}{27}-\frac{16\zeta_3}{3}.
$$

# Refactorization of the hemisphere calculation and its power corrections



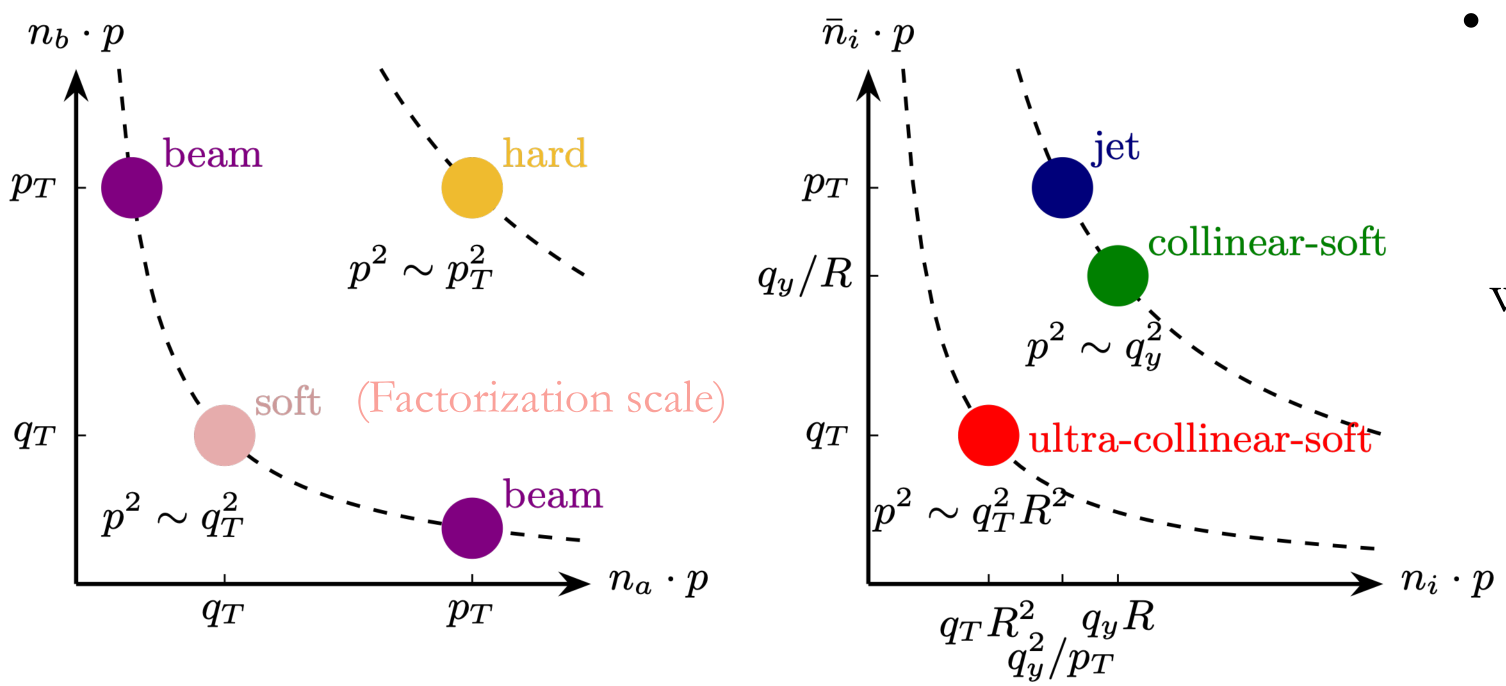




The comparison demonstrates that the two approaches are consistent in the small radius limit $R \ll 1$.

$$
\begin{aligned}
\bar{r}_A &= \bar{p}_A + R^{-2\epsilon}\bar{q}_A + \mathcal{O}(\epsilon^0 R\ln R, \epsilon^0 R) \\
&= \frac{2\pi^2}{3\epsilon^2} - \frac{2\left(3 - 11\pi^2 + 6\pi^2\ln 2 - 18\zeta_3\right)}{9\epsilon} + \Bigg[\left(\frac{4}{3} - \frac{44\pi^2}{9} + 8\zeta_3\right)\ln R \\
&\quad + \frac{40}{9} + \frac{7\pi^4}{15} + \frac{\pi^2}{27}\left(-134 + 36\ln^2 2\right) + \frac{44\zeta_3}{3} - 16\zeta_3\ln 2\Bigg] + \mathcal{O}(\epsilon^0 R\ln R, \epsilon^0 R),
\end{aligned}
$$

$$
\begin{aligned}
\bar{r}_f &= R^{-2\epsilon}\bar{q}_f + \mathcal{O}(\epsilon^0 R\ln R, \epsilon^0 R) \\
&= \left(\frac{4}{3} - \frac{8\pi^2}{9}\right)\frac{1}{\epsilon} + \frac{8}{9}\left(-3 + 2\pi^2\right)\ln R + \frac{4}{27}\left(-51 + 16\pi^2 - 36\zeta_3\right) + \mathcal{O}(\epsilon^0 R\ln R, \epsilon^0 R).
\end{aligned}
$$

# RRG and global RG evolutions




- Two sets of RRG evolution:

$$\prod_{i=ab}\left(\frac{\nu_s}{\nu_i}\right)^{\Gamma_\nu^i(b_T,\mu_f)} \qquad \prod_{i=12}\left(\frac{\nu_{cs,i}}{\nu_i}\right)^{\Gamma_\nu^i(b_y,\mu_{cs})}$$

with the rapidity scales

$$\nu_i \sim \omega_i \,(i = a, b, 1, 2), \quad \nu_s \sim b_0/b_T,$$

$$\nu_{cs,i} \sim b_0/(|b_y|R_i)\,(i = 1, 2).$$

- Typical scales in RG evolution: $\mu_h \sim 2p_T, \quad \mu_b \sim \mu_s \sim b_0/b_T, \quad \mu_{cs} \sim \mu_j \sim b_0/|b_y|, \quad \mu_{ucs} \sim R\, b_0/(2|b_x|)$.
- The RG evolution of the hard function is analogous to that for the $q_y$ distribution.
- The RG evolution of the jet function from $\mu_j$ to $\mu_f \sim b_0/b_T$ should be included to resum the large logarithms $\ln(|b_y|/b_T)$ that arise in the limit $\sin\phi_b \to 0$.

# Non-global RG evolution

- The evolution matrix for the collinear-soft and ultra-collinear-soft functions:

$$\prod_{i=1,2} \exp\left[\int_{\mu_{cs}}^{\mu_f} \frac{\mathrm{d}\mu}{\mu}\Gamma_i^{\mathrm{cs}} + \int_{\mu_{ucs}}^{\mu_f} \frac{\mathrm{d}\mu}{\mu}\Gamma_i^{\mathrm{ucs}}\right] U_{\mathrm{NG}}^i(\mu_{ucs}, \mu_{cs}).$$

- We resume the leading non-global logarithms $\ln R$,

$$U_{\mathrm{NG}}^i(\mu_{ucs}, \mu_{cs}) \overset{\mathrm{LL}_R}{=} \sum_{m=0}^{\infty} \left\langle \mathbf{1} \,\hat{\otimes}\, \boldsymbol{U}_{m0}(\{n_i, \bar{n}_i, \underline{u}\}, \mu_{ucs}, \mu_{cs}) \right\rangle$$

- Dasgupta–Salam parametrization [Dasgupta, Salam, '01]

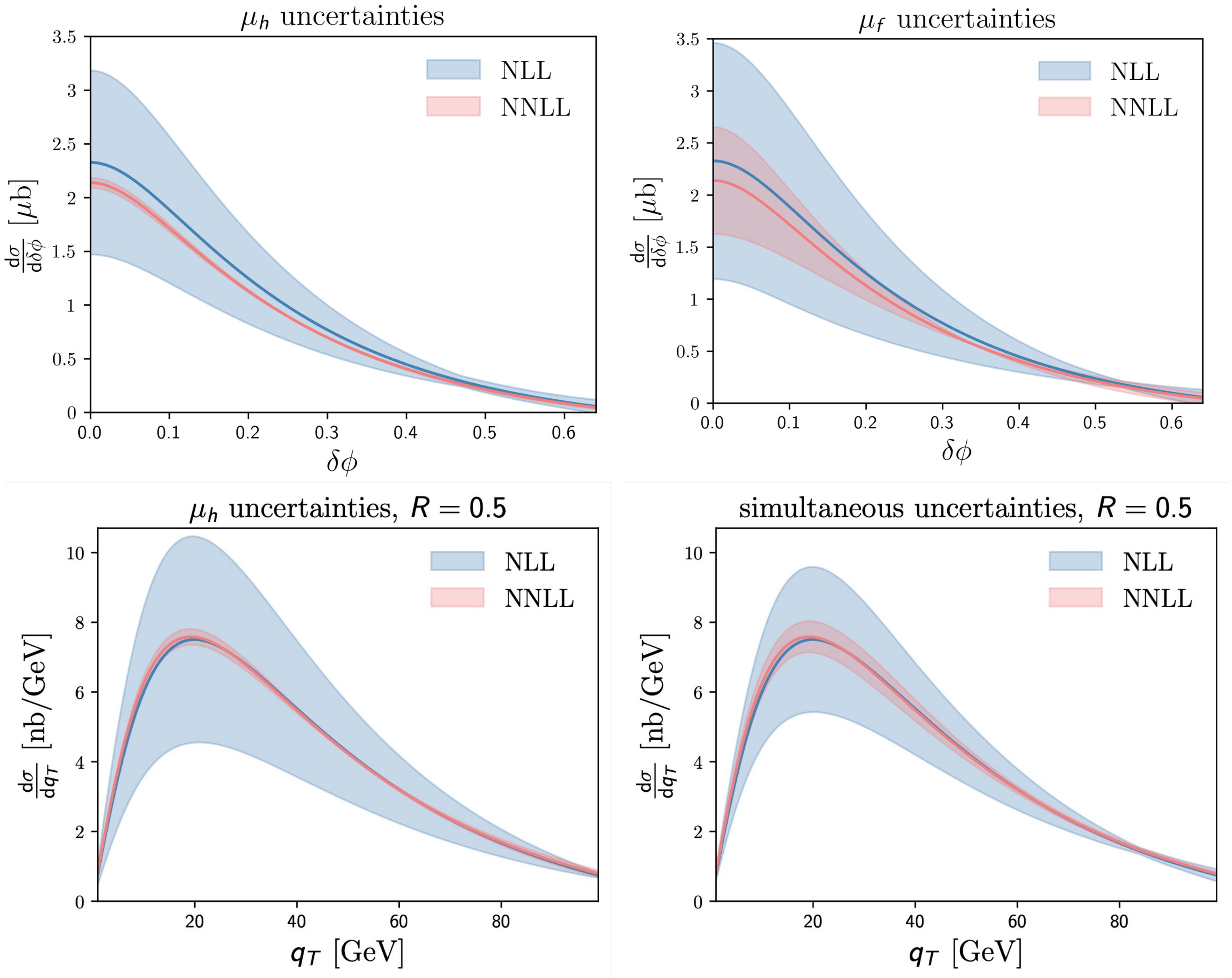
$$U_{\mathrm{NG}}^i(\mu_{ucs}, \mu_{cs}) \approx \exp\left(-C_A C_i \frac{\pi^2}{3} u^2 \frac{1+(au)^2}{1+(bu)^c}\right), \quad u = \frac{1}{\beta_0} \ln \frac{\alpha_s(\mu_{ucs})}{\alpha_s(\mu_{cs})},$$

with $a = 0.85 C_A, b = 0.86 C_A, c = 1.33.$

# NNLL+LL$_R$ Resummation formula for $q_T$

$$
\begin{aligned}
&\frac{\mathrm{d}^5\sigma}{\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}p_T\,\mathrm{d}q_T\,\mathrm{d}\phi_q} \\
&= \sum_{ijk\ell} \frac{x_a x_b}{16\pi \hat{s}^2} \frac{2p_T}{1+\delta_{k\ell}} q_T \int_0^{2\pi} \mathrm{d}\phi_b \int_0^\infty \frac{b_T\,\mathrm{d}b_T}{(2\pi)^2} e^{i\vec{q}_T\cdot\vec{b}_T} \\
&\times \sum_{KK'} \exp\left\{ \int_{\mu_h}^{\mu_f} \frac{\mathrm{d}\mu}{\mu} \left[ \gamma_{\mathrm{cusp}}(\alpha_s) \left( C_H \ln\frac{\hat{s}}{\mu^2} + \lambda_K + \lambda_{K'}^* \right) + 2\gamma_H(\alpha_s) \right] \right\} \\
&\times \prod_{p=ab} \left(\frac{\nu_s}{\nu_p}\right)^{\Gamma_\nu^p(b_T,\mu_f)} \prod_{p=12} \left(\frac{\nu_{cs,p}}{\nu_p}\right)^{\Gamma_\nu^p(b_y,\mu_{cs})} \\
&\times \prod_{p=12} \exp\left[ \int_{\mu_{cs}}^{\mu_f} \frac{\mathrm{d}\mu}{\mu} \left(\Gamma^{J_p} + \Gamma_p^{\mathrm{cs}}\right) + \int_{\mu_{ucs}}^{\mu_f} \frac{\mathrm{d}\mu}{\mu} \Gamma_p^{\mathrm{ucs}} \right] U_{\mathrm{NG}}^p(\mu_{ucs},\mu_{cs}) \\
&\times \boldsymbol{\mathcal{H}}_{ij\to k\ell,KK'}(p_T,\eta_1-\eta_2,\mu_h) \boldsymbol{\mathcal{S}}^{\mathrm{global}}_{ijk\ell,K'K}(\vec{b}_T,\eta_1,\eta_2,\mu_f,\nu_s) \\
&\times \prod_{p=12} \sum_{m_p=0}^{\infty} \left\langle \boldsymbol{\mathcal{S}}^{\mathrm{ucs}}_{m_p}(\{n_p,\bar{n}_p,\underline{u}\},\vec{b}_T,R,\mu_{ucs}) \otimes \boldsymbol{\mathcal{S}}^{\mathrm{cs}}_{m_p}(\{n_p,\bar{n}_p,\underline{u}\},b_y,R,\mu_{cs},\nu_{cs,p}) \right\rangle \\
&\times B_{i/p}(x_a,b_T,\omega_a,\mu_f,\nu_a)\, B_{j/p}(x_b,b_T,\omega_b,\mu_f,\nu_b)\, \mathcal{J}_k(b_y,\omega_1,\mu_j,\nu_1)\, \mathcal{J}_\ell(b_y,\omega_2,\mu_j,\nu_2) \\
&\times \exp\left[ -S_{\mathrm{NP}}^i(b_T,\omega_a,\nu_a) - S_{\mathrm{NP}}^j(b_T,\omega_b,\nu_b) \right].
\end{aligned}
$$

# NLL and NNLL Resummation results




- $b_*$-prescription:

$$b_{T,*} = \frac{b_T}{\sqrt{1 + b_T^2/b_{\max}^2}}, \quad b_{y,*} = \frac{|b_y|}{\sqrt{1 + b_y^2/b_{\max}^2}}, \quad b_{x,*} = \frac{|b_x|}{\sqrt{1 + b_x^2/b_{\max}^2}}$$

- Scales in $\delta\phi$ resummation:

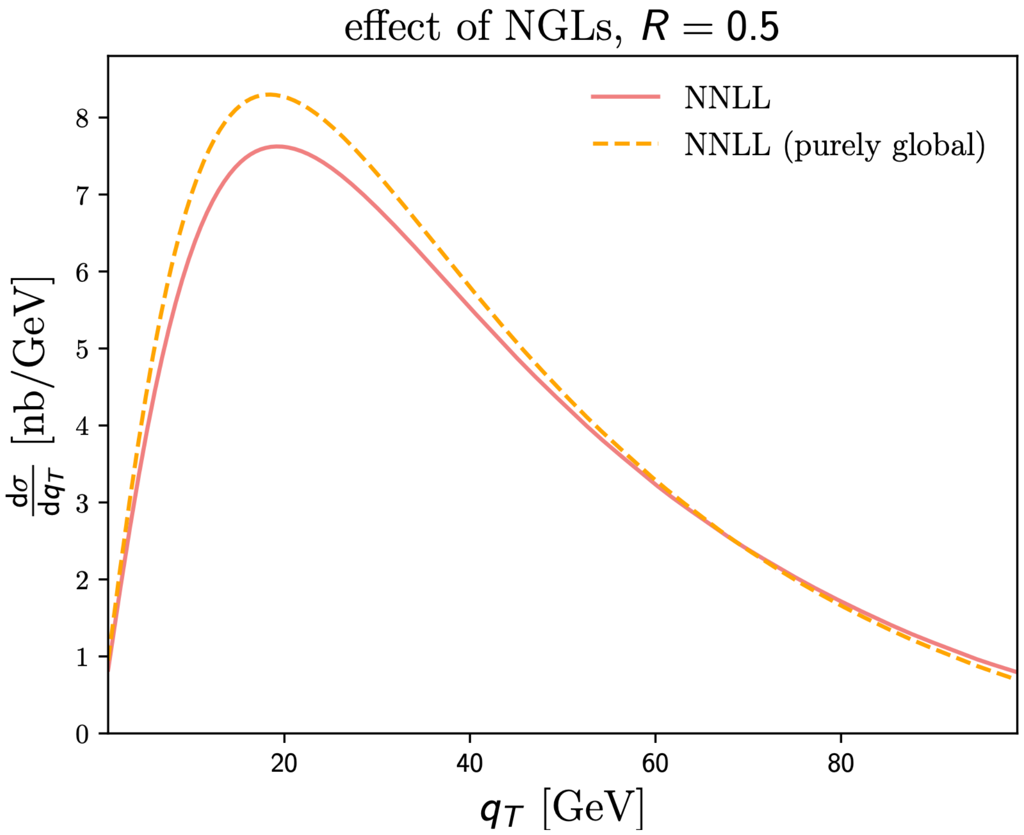
$$\mu_h = v_h \cdot 2p_T \qquad \mu_f = \min\{v_f \cdot \mu_{b_{y,*}}, \mu_h\}$$

- Scales in $q_T$ resummation:

$$\mu_s = \mu_b = \mu_f = \min\left\{v_f \frac{b_0}{b_{T,*}}, \mu_h\right\}, \quad \mu_j = \mu_{cs} = v_f \frac{b_0}{b_{y,*}},$$

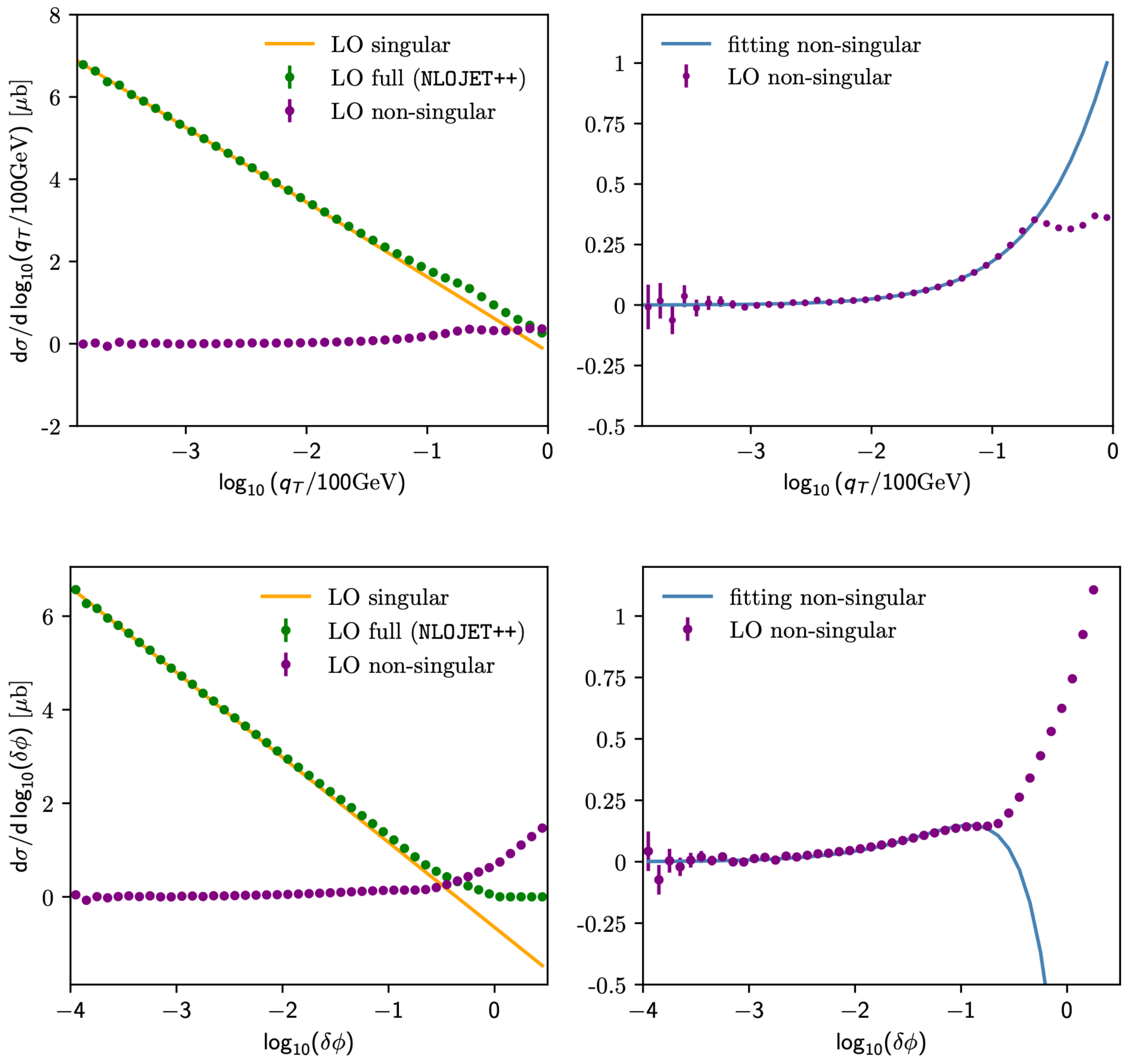
$$\mu_h = v_h \cdot 2p_T \qquad \mu_{ucs} = \min\left\{v_f \frac{Rb_0}{2b_{x,*}}, \mu_{cs}\right\}.$$

# Effects of NGLs




- The comparison reveals that the inclusion of the resummation of NGLs leads to a noticeable **suppression** of the cross-section in the peak region.

# Validation of factorization and Power corrections




- The non-singular contribution $(B_1 + B_0 \ln \mathcal{O})$ ($\mathcal{O} = \delta\phi$ or $q_T$) retains a logarithmic divergence as $\mathcal{O} \to 0$, despite being suppressed relative to the leading singular terms:

$$\frac{d\sigma(\text{LO full})}{d\log_{10}\mathcal{O}} \simeq \frac{d\sigma(\text{LO singular})}{d\log_{10}\mathcal{O}} + \frac{d\sigma(\text{power corrections})}{d\log_{10}\mathcal{O}}$$

$$= (A_1 \ln \mathcal{O} + A_0) + (B_1\mathcal{O} + B_0\mathcal{O}\ln\mathcal{O}),$$

$$\frac{d\sigma(\text{LO full})}{d\mathcal{O}} \simeq \frac{A_1 \ln \mathcal{O} + A_0}{\mathcal{O}} + (B_1 + B_0 \ln \mathcal{O}).$$

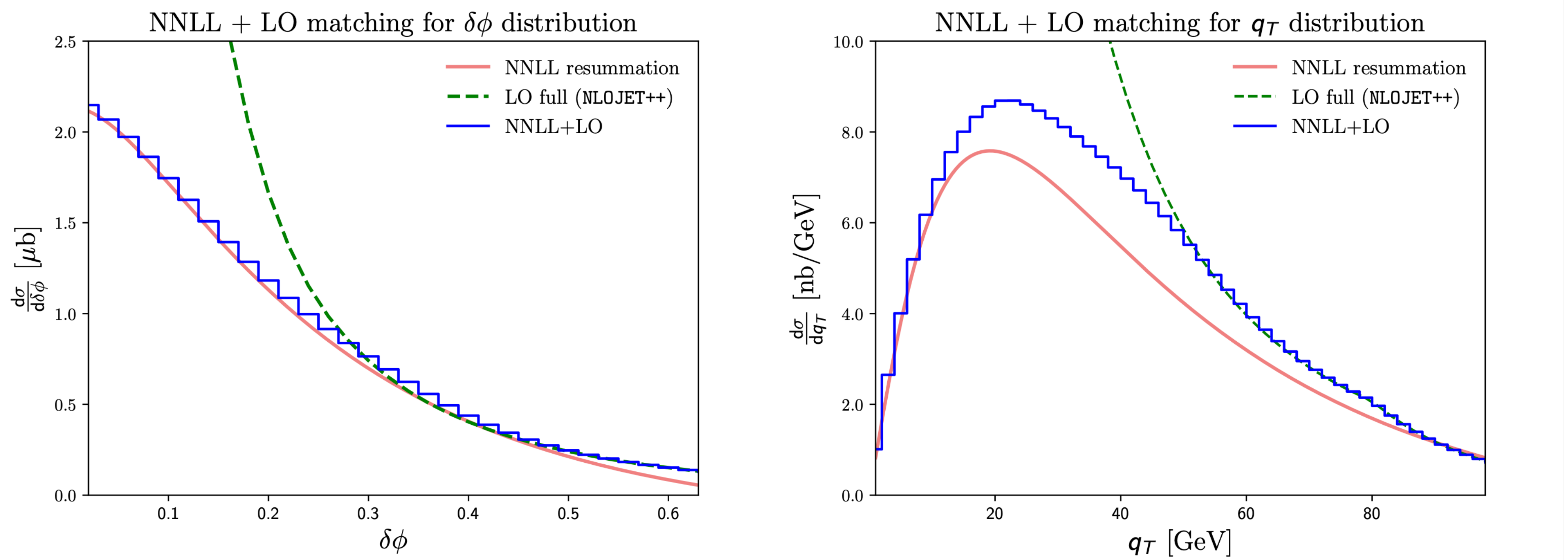
- This logarithmic enhancement is characteristic of exclusive or jet-dependent observables. [Ebert, Tackmann '19; Salam, Slade '21; Grazzini, Wiesemann '17]

# NNLL+LO Matching

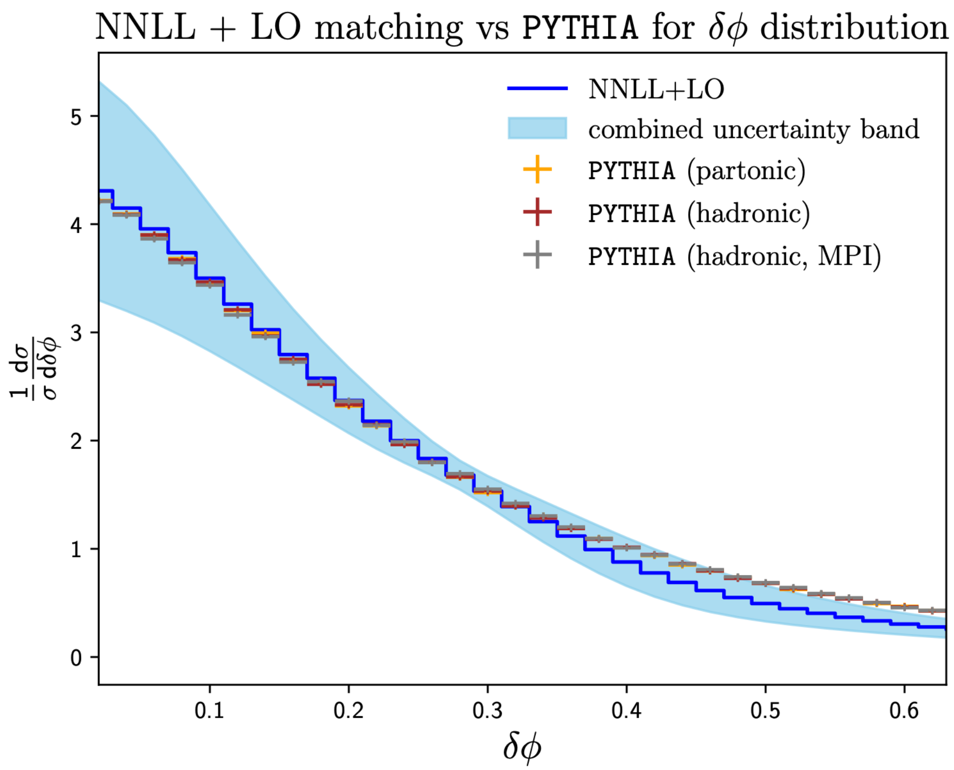
$$d\sigma_{\text{match}}(\text{NNLL}+\text{LO}) = (1-t(\mathcal{O}))\left(d\sigma(\text{NNLL}) + U_H^{\text{eff}}(\mathcal{O})\, d\sigma(\text{LO non-singular})\right) + t(\mathcal{O})\, d\sigma(\text{LO full}).$$

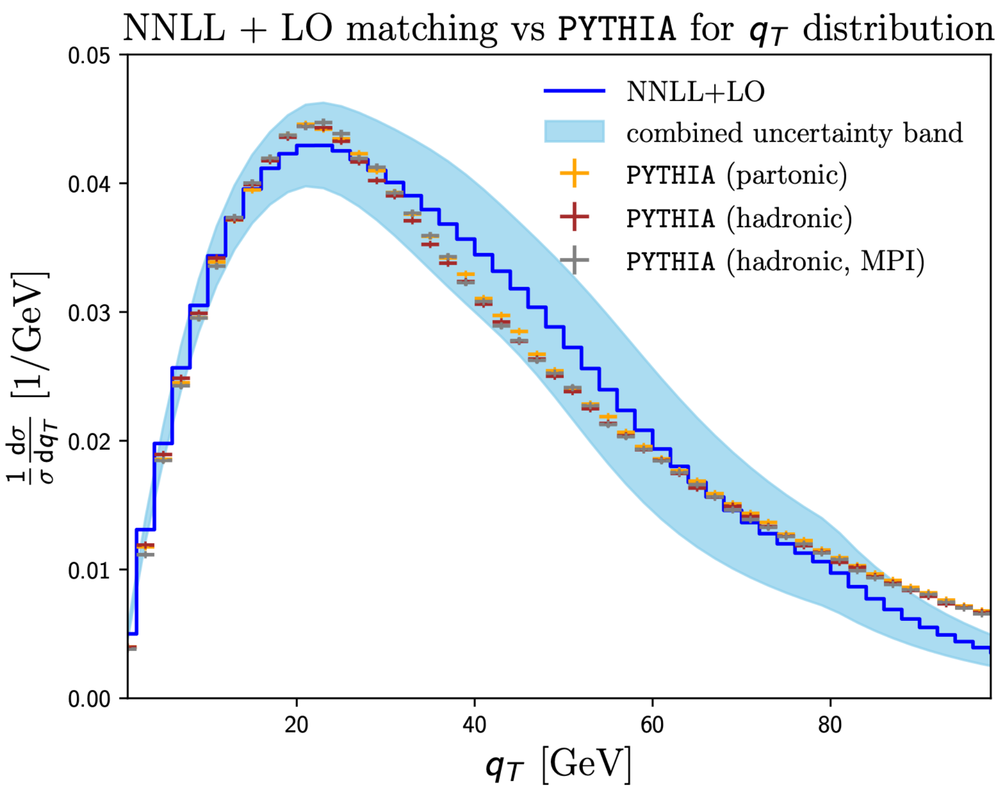
- The effective evolution matrix, $U_H^{\text{eff}}(\mathcal{O}) = \exp\left[\int_{2p_T}^{\mu_{\mathcal{O}}} \frac{d\mu}{\mu}\left(\frac{\alpha_s(2p_T)}{\pi} 4C_A \ln\frac{4p_T^2}{\mu^2}\right)\right]$,

  is used to suppress the divergent non-singular terms in the infrared regime.
- The transition function $t(\mathcal{O})$ acts as a smooth switch, transitioning from 0 in the resummation-dominated region to 1 in the fixed-order region.



# Comparison with PYTHIA 8.3




- Both the azimuthal decorrelation $\delta\phi$ and the transverse momentum imbalance $q_T$ are robust against hadronization and multi-parton interactions (MPI).

# Conclusions

- Novel Slicing Framework for Jets: Proposed winner-take-all (WTA) azimuthal decorrelation $\delta\phi$ (planar) and transverse momentum imbalance $q_T$ (general) slicing variables to handle IR divergences.

- Resummation for Dijet Production ($pp \to 2$jets):
  - WTA scheme eliminates standard non-global logarithms (NGLs), simplifying the factorization;
  - Small-$R$ Limit: Exploiting $R \ll 1$ allows the soft function to be **refactorized** into global, collinear-soft, and ultra-collinear-soft modes, enabling the calculation of NGLs of $R$;
  - Joint resummation involving NNLL accuracy for global logarithms and LL accuracy for the small-$R$ NGLs.

# THANK YOU!

# Back-up: Anomalous dimensions

- The standard form of the RG equations of collinear-soft and ultra-collinear-soft functions can be derived as,

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\boldsymbol{\mathcal{S}}_m^{\mathrm{cs}}(\{n_i,\bar{n}_i,\underline{u}\},b_y,R,\mu,\nu) = \sum_{l=0}^{m}\boldsymbol{\Gamma}_{ml}^{\mathrm{cs}}\,\boldsymbol{\mathcal{S}}_l^{\mathrm{cs}}(\{n_i,\bar{n}_i,\underline{u}\},b_y,R,\mu,\nu)$$
$$= \sum_{l=0}^{m}\left(\Gamma^{\mathrm{cs}}\delta_{ml}\mathbf{1}+\hat{\boldsymbol{\Gamma}}_{ml}\right)\boldsymbol{\mathcal{S}}_l^{\mathrm{cs}}(\{n_i,\bar{n}_i,\underline{u}\},b_y,R,\mu,\nu),$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\boldsymbol{\mathcal{S}}_l^{\mathrm{ucs}}(\{n_i,\bar{n}_i,\underline{u}\},b_x,R,\mu) = -\sum_{m=l}^{\infty}\boldsymbol{\mathcal{S}}_m^{\mathrm{ucs}}(\{n_i,\bar{n}_i,\underline{u}\},b_x,R,\mu)\hat{\otimes}\boldsymbol{\Gamma}_{ml}^{\mathrm{ucs}}$$
$$= \sum_{m=l}^{\infty}\boldsymbol{\mathcal{S}}_m^{\mathrm{ucs}}(\{n_i,\bar{n}_i,\underline{u}\},b_x,R,\mu)\hat{\otimes}\left(\Gamma^{\mathrm{ucs}}\delta_{ml}\mathbf{1}-\hat{\boldsymbol{\Gamma}}_{ml}\right).$$

- Their combination $S_i$, evolves under the RG without any residual multiplicity mixing,

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}S_i(\vec{b}_T,\eta_i,R,\mu,\nu) = \left(\Gamma^{\mathrm{cs}}+\Gamma^{\mathrm{ucs}}\right)S_i(\vec{b}_T,\eta_i,R,\mu,\nu).$$

- The two-loop global anomalous dimensions for the collinear-soft and ultra-collinear-soft functions are

$$\Gamma^{\mathrm{cs}}(\alpha_s,\{n_i,\bar{n}_i\},R,\mu,\nu) = 2C_i\gamma_{\mathrm{cusp}}(\alpha_s)\ln\frac{\mu}{\nu R_i},$$
$$\Gamma^{\mathrm{ucs}}(\alpha_s,\{n_i,\bar{n}_i\},b_x,R,\mu) = -C_i\gamma_{\mathrm{cusp}}(\alpha_s)\left[\ln\frac{4\mu^2 b_x^2}{b_0^2R^2}-i\pi\mathrm{Sign}(b_x n_{i,x})\right],$$